2. FUNCTIONS

FUNCTION *Let A and B be two nonempty sets. Then, a rule f which associates to each element $x \in A$, a unique element, denoted by $f(x)$ of B, is called a function from A to B and we write,*

$$f : A \in B.$$

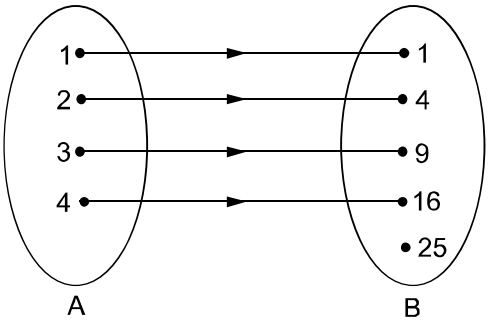
*$f(x)$ is called the **image** of x, while x is called the **pre-image** of $f(x)$.*

Domain, Codomain and Range of a Function

Let $f : A \to B$. Then, A is called the *domain* of f and B is called the *codomain* of f.

And, $f(A) = \{f(x) : x \in A\}$ is called the *range* of f.

Example 1 Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$.



Consider the rule $f : A \to B : f(x) = x^2 \ \forall \ x \in A$.

Then, each element in A has its unique image in B.

So, f is a function from A to B.

$$f(1) = 1^2 = 1, \ \ f(2) = 2^2 = 4, \ \ f(3) = 3^2 = 9, \ \ f(4) = 4^2 = 16.$$

Dom $(f) = \{1, 2, 3, 4\} = A$, codomain $(f) = \{1, 4, 9, 16, 25\} = B$ and range $(f) = \{1, 4, 9, 16\}$.

Clearly, $25 \in B$ does not have its pre-image in A.

Example 2 Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \ \forall \ x \in N$.

Then, every element in N has its unique image in N.

So, f is a function from N to N.

Clearly, $f(1) = 2, \ \ f(2) = 4, \ \ f(3) = 6$ ..., and so on.

Dom $(f) = N$, codomain $(f) = N$,

range $(f) = \{2, 4, 6, 8, 10, ...\}$.

Various Types of Functions

MANY-ONE FUNCTION *A function $f : A \to B$ is said to be many-one if two or more than two elements in A have the same image in B.*

Example Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$.

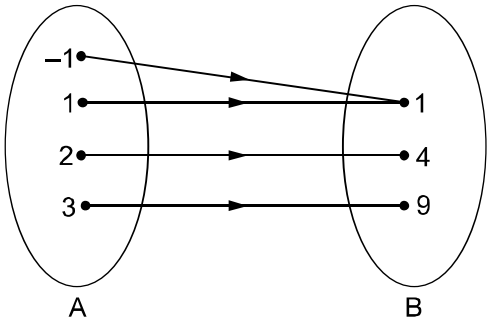
Let $f : A \to B : f(x) = x^2 \ \forall \ x \in A$.

Then, each element in A has a unique image under f in B.

$\therefore$ f is a function from A to B such that

$f(-1) = (-1)^2 = 1; \ f(1) = 1^2 = 1;$

$f(2) = 2^2 = 4$ and $f(3) = 3^2 = 9$.



Clearly, two different elements, namely –1 and 1, have the same image $1 \in B$.

Hence, f is many-one.

One-One or Injective Function

A function $f : A \to B$ is said to be one-one if distinct elements in A have distinct images in B.

f is one-one when $f(x_1) = f(x_2) \ \Rightarrow \ x_1 = x_2$.

Example Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \ \forall \ x \in N$.

Then, $f(x_1) = f(x_2) \ \Rightarrow \ 2x_1 = 2x_2$

$\Rightarrow \ x_1 = x_2$.

$\therefore$ f is one-one.

Onto or Surjective Function

A function $f : A \to B$ is said to be onto if every element in B has at least one pre-image in A.

Thus, if f is onto then for each $y \in B$ $\exists$ at least one element $x \in A$ such that $y = f(x)$.

Also, f is onto $\Leftrightarrow$ *range* $(f) = B$.

Example Let N be the set of all natural numbers and let E be the set of all even natural numbers.

Let $f : N \to E : f(x) = 2x \ \forall \ x \in N$.

Then, $y = 2x \ \Rightarrow \ x = \frac{1}{2} y$.

Thus, for each $y \in E$ there exists $\frac{1}{2}y \in N$ such that

$$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

$\therefore$ f is onto.

INTO FUNCTION *A function $f : A \to B$ is said to be into if there exists even a single element in B having no pre-image in A.*

Clearly, f is into $\Leftrightarrow$ range $(f) \subset B$.

Example Let $A = \{2, 3, 5, 7\}$ and $B = \{0, 1, 3, 5, 7\}$.

Let $f : A \to B : f(x) = (x - 2)$. Then,

$f(2) = (2 - 2) = 0$, $f(3) = (3 - 2) = 1$, $f(5) = (5 - 2) = 3$ and $f(7) = (7 - 2) = 5$.

Thus, every element in A has a unique image in B.

Now, $\exists\ 7 \in B$ having no pre-image in A.

$\therefore$ f is into.

Note that range $(f) = \{0, 1, 3, 5\} \subset B$.

Bijective Function

A one-one onto function is said to be bijective or a one-to-one correspondence.

CONSTANT FUNCTION *A function $f : A \to B$ is called a constant function if every element of A has the same image in B.*

Example Let $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$.

Let $f : A \to B : f(x) = 5$ for all $x \in A$.

Clearly, every element in A has the same image.

So, f is a constant function.

REMARK The range of a constant function is a singleton set.

IDENTITY FUNCTION *The function $I_A : A \to A : I_A(x) = x\ \forall\ x \in A$ is called an identity function on A.*

Domain $(I_A) = A$ and range $(I_A) = A$.

EQUAL FUNCTIONS *Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)\ \forall\ x \in D$.*

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : N \to N : f(x) = 2x$ for all $x \in N$.*
Show that f is one-one and into.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Let $y = 2x$. Then, $x = \frac{y}{2}$.

If we put $y = 3$ then $x = \frac{3}{2} \notin N$.

Thus, $3 \in N$ has no pre-image in N.

$\therefore$ f is into.

Hence, f is one-one and into.

EXAMPLE 2 *Show that the function* $f : R \to R : f(x) = x^2$ *is neither one-one nor onto.*

SOLUTION We have $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider –1 in the codomain R, then it is clear that there is no element in R whose image is -1.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 3 *Show that the function* $f : R \to R : f(x) = |x|$ *is neither one-one nor onto.*

SOLUTION We have $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider –1 in the codomain R, then it is clear that there is no real number x whose modulus is –1.

Thus, $-1 \in R$ has no pre-image in R.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 4 *For any real number x, we define*

$[x]$ = *greatest integer less than or equal to x.*

Prove that the greatest integer function $f : R \to R : f(x) = [x]$ *is neither one-one nor onto.*

SOLUTION Clearly, $[1.2] = 1$ and $[1.3] = 1$.

$\therefore$ $f(1.2) = 1$ and $f(1.3) = 1$.

Thus, two different real numbers have the same image.

$\therefore$ f is not one-one.

Clearly, there is no real number x such that

$f(x) = [x] = 1.1$.

So, f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 5 *Let R_0 be the set of all nonzero real numbers.*
Show that $f : R_0 \to R_0 : f(x) = \dfrac{1}{x}$ is a one-one onto function.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Again, $y = \dfrac{1}{x} \Rightarrow x = \dfrac{1}{y}$.

Now, if y is a nonzero real number, then $x = \left(\dfrac{1}{y}\right)$ is a nonzero real number such that $f\left(\dfrac{1}{y}\right) = y$.

Thus, each y in R_0 has its pre-image in R_0.
So, f is onto.
Hence, f is one-one onto.

EXAMPLE 6 *Show that the function $f : R \to R : f(x) = x^3$ is one-one and onto.*

SOLUTION We have

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^3 = x_2^3 \\ &\Rightarrow (x_1^3 - x_2^3) = 0 \\ &\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \\ &\Rightarrow (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2\right] = 0 \\ &\Rightarrow (x_1 - x_2) = 0 \quad \left[\because \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 \neq 0\right] \\ &\Rightarrow x_1 = x_2. \end{aligned}$$

$\therefore$ f is one-one.

Let $y \in R$ and let $y = x^3$. Then, $x = y^{1/3} \in R$.

Thus, for each y in the codomain R there exists $y^{1/3}$ in R such that $f(y^{1/3}) = (y^{1/3})^3 = y$.

$\therefore$ f is onto.
Hence, f is one-one onto.

EXAMPLE 7 *Show that the function $f : R \to R : f(x) = 3 - 4x$ is one-one onto and hence bijective.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 3 - 4x_1 = 3 - 4x_2$$

$\Rightarrow \quad -4x_1 = -4x_2 \Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

Now, let $y = 3 - 4x$. Then, $x = \dfrac{(3-y)}{4}$.

Thus, for each $y \in R$ (codomain of f), there exists $x = \dfrac{(3-y)}{4} \in R$ such that $f(x) = f\left(\dfrac{3-y}{4}\right) = \left\{3 - 4 \cdot \dfrac{(3-y)}{4}\right\} = 3 - (3-y) = y.$

This shows that every element in codomain of f has its pre-image in dom (f).

$\therefore \quad f$ is onto.

Hence, the given function is bijective.

EXAMPLE 8 *Show that the function $f : N \to N$, defined by*

$$f(x) = \begin{cases} x + 1, & \text{if } x \text{ is odd} \\ x - 1, & \text{if } x \text{ is even} \end{cases}$$

is one-one and onto. **[CBSE 2012]**

SOLUTION Suppose $f(x_1) = f(x_2)$.

Case 1 *When x_1 is odd and x_2 is even*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2.$

This is a contradiction, since the difference between an odd integer and an even integer can never be 2.

Thus, in this case, $f(x_1) \neq f(x_2)$.

Similarly, when x_1 is even and x_2 is odd, then $f(x_1) \neq f(x_2)$.

Case 2 *When x_1 and x_2 are both odd*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

Case 3 *When x_1 and x_2 are both even*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

In order to show that f is onto, let $y \in N$ (the codomain).

Case 1 *When y is odd*

In this case, $(y + 1)$ is even.

$\therefore \quad f(y + 1) = (y + 1) - 1 = y.$

Case 2 *When y is even*

In this case, $(y - 1)$ is odd.

$\therefore \quad f(y - 1) = y - 1 + 1 = y.$

Thus, each $y \in N$ (codomain of f) has its pre-image in dom (f).

$\therefore$ f is onto.

Hence, f is one-one onto.

EXAMPLE 9 *Show that* $f : N \to N$, *defined by*

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

is a many-one onto function. **[CBSE 2012C]**

SOLUTION We have

$$f(1) = \frac{(1+1)}{2} = \frac{2}{2} = 1 \quad \text{and} \quad f(2) = \frac{2}{2} = 1.$$

Thus, $f(1) = f(2)$ while $1 \neq 2$.

$\therefore$ f is many-one.

In order to show that f is onto, consider an arbitrary element $n \in N$.

If n is odd then $(2n-1)$ is odd, and

$$f(2n-1) = \frac{(2n-1+1)}{2} = \frac{2n}{2} = n.$$

If n is even then $2n$ is even and

$$f(2n) = \frac{2n}{2} = n.$$

Thus, for each $n \in N$ (whether even or odd) there exists its pre-image in N.

$\therefore$ f is onto.

Hence, f is many-one onto.

EXAMPLE 10 *Show that the signum function* $f : R \to R$, *defined by*

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

SOLUTION Clearly, $f(2) = 1$ and $f(3) = 1$.

Thus, $f(2) = f(3)$ while $2 \neq 3$.

$\therefore$ f is not one-one.

Range (f) $= \{1, \ 0, \ -1\} \subset R$.

So, f is into.

Hence, f is neither one-one nor onto.

EXAMPLE 11 *Let* $A = R - \{3\}$ *and* $B = R - \{1\}$.

Let $f : A \to B : f(x) = \dfrac{x-2}{x-3}$ *for all values of* $x \in A$.

Show that f is one-one and onto. **[CBSE 2012]**

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$

$$\Rightarrow x_1 = x_2.$$

Let $y \in B$ such that $y = \dfrac{x-2}{x-3}$.

Then, $(x-3)\ y = (x-2) \Rightarrow x = \dfrac{(3y-2)}{(y-1)}$.

Clearly, x is defined when $y \neq 1$.

Also, $x = 3$ will give us $1 = 0$, which is false.

$\therefore\ x \neq 3$.

And, $f(x) = \dfrac{\left(\dfrac{3y-2}{y-1} - 2\right)}{\left(\dfrac{3y-2}{y-1} - 3\right)} = y.$

Thus, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

$\therefore\ f$ is onto.

Hence, f is one-one onto.

EXAMPLE 12 *Let A and B be two nonempty sets. Show that the function* $f : (A \times B) \to (B \times A) : f(a,\ b) = (b,\ a)$ *is a bijective function.*

SOLUTION f is one-one, since

$$f(a_1,\ b_1) = f(a_2,\ b_2) \Rightarrow (b_1,\ a_1) = (b_2,\ a_2)$$

$$\Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$

$$\Rightarrow (a_1,\ b_1) = (a_2,\ b_2).$$

In order to show that f is onto, let $(b,\ a)$ be an arbitrary element of $(B \times A)$.

Then, $(b,\ a) \in (B \times A) \Rightarrow b \in B$ and $a \in A$

$$\Rightarrow (a,\ b) \in (A \times B).$$

Thus, for each $(b,\ a) \in (B \times A)$, there exists $(a,\ b) \in A \times B$ such that $f(a,\ b) = (b,\ a)$.

$\therefore\ f$ is onto.

Thus, f is one-one onto and hence bijective.

EXAMPLE 13 *Find the domain and range of the real function*

$$f(x) = \sqrt{9 - x^2}.$$

SOLUTION It is clear that $f(x) = \sqrt{9 - x^2}$ is not defined when $(9 - x^2) < 0$, i.e., when $x^2 > 9$, i.e., when $x > 3$ or $x < -3$.

$\therefore$ dom $(f) = \{x \in R : -3 \le x \le 3\}$.

Also, $y = \sqrt{9 - x^2} \Rightarrow y^2 = (9 - x^2)$

$\Rightarrow x = \sqrt{9 - y^2}$.

Clearly, x is not defined when $(9 - y^2) < 0$.

But, $(9 - y^2) < 0 \Rightarrow y^2 > 9$

$\Rightarrow y > 3$ or $y < -3$

$\therefore$ range $(f) = \{y \in R : -3 \le y \le 3\}$.

EXAMPLE 14 *Find the domain and range of the real function, defined by*

$$f(x) = \frac{1}{(1 - x^2)}.$$

SOLUTION Clearly, $\frac{1}{(1 - x^2)}$ is not defined when $1 - x^2 = 0$,

i.e., when $x = \pm 1$.

$\therefore$ dom $(f) = R - \{-1, 1\}$.

Also, $y = \frac{1}{(1 - x^2)} \Rightarrow (1 - x^2) = \frac{1}{y} \Rightarrow x = \sqrt{1 - \frac{1}{y}}$.

Clearly, x is not defined when $\left(1 - \frac{1}{y}\right) < 0$ or $1 < \frac{1}{y}$ or $y < 1$.

$\therefore$ range $(f) = R - \{y \in R : y < 1\} = \{y \in R : y \ge 1\}$.

EXAMPLE 15 *Consider a function $f : X \to Y$ and define a relation R in X by $R = \{(a, b) : f(a) = f(b)\}$. Show that R is an equivalence relation.*

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*

Let $a \in X$. Then,

$f(a) = f(a) \Rightarrow (a, a) \in R$.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$.

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R, (b, c) \in R$

$\Rightarrow f(a) = f(b)$ and $f(b) = f(c)$

$\Rightarrow f(a) = f(c)$

$\Rightarrow (a, c) \in R.$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

EXERCISE 2A

1. Define a function. What do you mean by the domain and range of a function? Give examples.
2. Define each of the following:
   (i) injective function (ii) surjective function
   (iii) bijective function (iv) many-one function
   (v) into function
   Give an example of each type of functions.
3. Give an example of a function which is
   (i) one-one but not onto (ii) one-one and onto
   (iii) neither one-one nor onto (iv) onto but not one-one.
4. Let $f : R \to R$ be defined by
$$f(x) = \begin{cases} 2x + 3, & \text{when } x < -2 \\ x^2 - 2, & \text{when } -2 \le x \le 3 \\ 3x - 1, & \text{when } x > 3. \end{cases}$$
   Find (i) $f(2)$ (ii) $f(4)$ (iii) $f(-1)$ (iv) $f(-3)$.
5. Show that the function $f : R \to R : f(x) = 1 + x^2$ is many-one into.
6. Show that the function $f : R \to R : f(x) = x^4$ is many-one and into.
7. Show that the function $f : R \to R : f(x) = x^5$ is one-one and onto.
8. Let $f : \left[0, \frac{\pi}{2}\right] \to R : f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to R : g(x) = \cos x$. Show that each one of f and g is one-one but $(f + g)$ is not one-one.
9. Show that the function
   (i) $f : N \to N : f(x) = x^2$ is one-one into.
   (ii) $f : Z \to Z : f(x) = x^2$ is many-one into.

10. Show that the function
    (i) $f : N \to N : f(x) = x^3$ is one-one into
    (ii) $f : Z \to Z : f(x) = x^3$ is one-one into

11. Show that the function $f : R \to R : f(x) = \sin x$ is neither one-one nor onto.

12. Prove that the function $f : N \to N : f(n) = (n^2 + n + 1)$ is one-one but not onto.

13. Show that the function $f : N \to Z$, defined by
$$f(n) = \begin{cases} \frac{1}{2}(n-1), & \text{when } n \text{ is odd} \\ -\frac{1}{2}n, & \text{when } n \text{ is even} \end{cases}$$
is both one-one and onto.

14. Find the domain and range of the function
$f : R \to R : f(x) = x^2 + 1.$

15. Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.
    (i) $f = \{(-1, 2), (1, 8), (2, 11), (3, 14)\}$
    (ii) $g = \{(1, 1), (1, -1), (4, 2), (9, 3), (16, 4)\}$
    (iii) $h = \{(a, b), (b, c), (c, b), (d, c)\}$

16. Find the domain and range of the real function, defined by $f(x) = \frac{x^2}{(1 + x^2)}$.
Show that f is many-one.

17. Show that the function
$$f : R \to R : f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$
is many-one into.
Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f(\sqrt{2})$ (iii) $f(\pi)$ (iv) $f(2 + \sqrt{3})$.

ANSWERS (EXERCISE 2A)

4. (i) 2 (ii) 11 (iii) –1 (iv) –3

14. dom $(f) = R$ and range $(f) = \{y \in R : y \geq 1\}$

15. (i) f is a function, dom $(f) = \{-1, 1, 2, 3\}$ and range $(f) = \{2, 8, 11, 14\}$
(ii) g is not a function
(iii) h is a function, dom $(h) = \{a, b, c, d\}$ and range $(h) = \{b, c\}$

17. (i) 1 (ii) –1 (iii) –1 (iv) –1

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2A)

2. (i) $f: N \to N: f(x) = 2x$ is an injective function, as
$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

(ii) Let $A = \{1, -1, 2, 3\}$ and $B = \{1, 4, 9\}$.
Then, $f: A \to B: f(x) = x^2$ is surjective, since each element of B has at least one pre-image in A.

(iii) Let E be the set of all even natural numbers.
Then, $f: N \to E: f(x) = 2x$ is one-one and onto.
f is one-one, since
$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
f is onto, since for each $y \in E$, there exists $\frac{1}{2}y \in N$ such that $f\left(\frac{1}{2}y\right) = y$.

(iv) Example given in (ii) is many-one.

(v) Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9, 16\}$.
Then, $f: A \to B: f(x) = x^2$ is an into function, since range $(f) = \{1, 4, 9\} \subset B$.

5. $f(-1) = 2 = f(1)$. So, f is many-one.
$-1 \in R$ has no pre-image in R. So, f is into.

6. $f(-1) = f(1) = 1$. So, f is many-one.
$-1 \in R$ has no pre-image in R. So, f is into.

7. $f(x_1) = f(x_2) \Rightarrow x_1^5 = x_2^5 \Rightarrow x_1 = x_2$. So, f is one-one
for each $y \in R \; \exists \; y^{1/5} \in R$ s.t. $f(y^{1/5}) = y$.

8. If $x_1 \neq x_2$ and $x_1, x_2 \in \left[0, \frac{\pi}{2}\right]$ then $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$.
$\therefore$ f is one-one and g is one-one.
But $(f + g)(x) = f(x) + g(x) = \sin x + \cos x$.
$\therefore$ $(f + g)(0) + (\sin 0 + \cos 0) = 1$ and $(f + g)\left(\frac{\pi}{2}\right) = \left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) = 1$.

9. (i) $f(a) = f(b) \Rightarrow a^2 = b^2 \Rightarrow a = b$ $[\because a, b \in N]$
$\therefore$ f is one-one.
Clearly, $2 \in N$ [codomain (f)] has no pre-image in N.
$\therefore$ f is into.

(ii) $f(-1) = f(1) = 1$. So, f is many-one.
$2 \in N$ [codomain (f)] has no pre-image in N.
$\therefore$ f is into.

10. (i) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ $[\because x_1, x_2 \in N]$.
$\therefore$ f is one-one.
$2 \in N$. But, $2^{1/3} \notin N$. So, f is into.

(ii) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ $[\because x_1, x_2 \in Z]$
$2 \in Z$. But, $2^{1/3} \notin Z$. So, f is into.

11. We know that $\sin(0) = 0$ and $\sin(\pi) = 0$.

Thus, 0 and π have the same image.

So, f is many-one.

Range $(f) = [-1,\ 1] \subset R$. Hence, f is into.

So, f is neither one-one nor onto.

12. $f(n_1) = f(n_2) \Rightarrow n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1$

$\Rightarrow (n_1^2 - n_2^2) + (n_1 - n_2) = 0$

$\Rightarrow (n_1 - n_2)(n_1 + n_2 + 1) = 0$

$\Rightarrow n_1 - n_2 = 0 \Rightarrow n_1 = n_2$.

$\therefore$ f is one-one.

But, $f(n) = 1 \Rightarrow n^2 + n + 1 = 1 \Rightarrow n^2 + n = 0$

$\Rightarrow n(n + 1) = 0 \Rightarrow n = 0$ or $n = -1$.

And, none of 0 and –1 is a natural number.

Thus, $1 \in N$ has no pre-image in N.

$\therefore$ f is into.

14. Dom $(f) = R$. Also, $y = x^2 + 1 \Rightarrow x = \sqrt{y - 1}$.

x is defined when $y - 1 \geq 0$, i.e., $y \geq 1$.

$\therefore$ range $(f) = \{y \in R : y \geq 1\}$.

15. g is not a function, since 1 has two images under g.

16. When x is real, $1 + x^2 \neq 0$. So, dom $(f) = R$.

$$y = \frac{x^2}{(1 + x^2)} \Rightarrow x^2(1 - y) = y \Rightarrow x = \sqrt{\frac{y}{1 - y}}.$$

For x to be real, $\frac{y}{(1 - y)} \geq 0$ and $(1 - y) \neq 0$.

$\therefore$ range $(f) = \{y \in R : 0 \leq y < 1\}$.

Also, 1 and –1 have the same image $\left(\frac{1}{2}\right)$.

Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two given functions. Then, the composition of f and g, denoted by g o f is the function, defined by

$$(g \text{ o } f) : A \to C : (g \text{ o } f)(x) = g\{f(x)\} \ \forall \ x \in A.$$

Clearly, dom $(g$ o $f) =$ dom (f).

Also, g o f is defined only when range $(f) \subseteq$ dom (g).

REMARK $(f$ o $g)$ is defined only when range $(g) \subseteq$ dom (f).
And, dom $(f$ o $g) =$ dom (g).

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f : \{1, 3, 4\} \to \{1, 2, 5\}$ *and* $g : \{1, 2, 5\} \to \{1, 3\}$ *be defined as* $f = \{(1, 2), (3, 5), (4, 1)\}$ *and* $g = \{(1, 3), (2, 3), (5, 1)\}$. *Find* $(g \circ f)$ *and* $(f \circ g)$.

SOLUTION Here range $(f) = \{1, 2, 5\}$ and dom $(g) = \{1, 2, 5\}$.

Clearly, range $(f) \subseteq$ dom (g).

$\therefore$ $(g \circ f)$ is defined and dom $(g \circ f) =$ dom $(f) = \{1, 3, 4\}$.

Now, $(g \circ f)(1) = g\{f(1)\} = g(2) = 3$;

$(g \circ f)(3) = g\{f(3)\} = g(5) = 1$;

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$.

Hence, $(g \circ f) = \{(1, 3), (3, 1), (4, 3)\}$.

Again, range $(g) = \{1, 3\}$ and dom $(f) = \{1, 3, 4\}$.

Clearly, range $(g) \subseteq$ dom (f).

$\therefore$ $(f \circ g)$ is defined and dom $(f \circ g) =$ dom $(g) = \{1, 2, 5\}$.

Now, $(f \circ g)(1) = f\{g(1)\} = f(3) = 5$;

$(f \circ g)(2) = f\{g(2)\} = f(3) = 5$;

$(f \circ g)(5) = f\{g(5)\} = f(1) = 2$.

Hence, $(f \circ g) = \{(1, 5), (2, 5), (5, 2)\}$.

EXAMPLE 2 *Let R be the set of all real numbers. Let* $f : R \to R : f(x) = \cos x$ *and let* $g : R \to R : g(x) = 3x^2$. *Show that* $(g \circ f) \neq (f \circ g)$.

SOLUTION Let x be an arbitrary real number. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x.$$

$$(f \circ g)(x) = f\{g(x)\} = f(3x^2) = \cos(3x^2).$$

Taking $x = 0$, we have

$$(g \circ f)(0) = 3\cos^2 0 = (3 \times 1) = 3.$$

$$(f \circ g)(0) = \cos(3 \times 0) = \cos 0 = 1.$$

$\therefore$ $(g \circ f)(0) \neq (f \circ g)(0)$.

Hence, $g \circ f \neq f \circ g$.

EXAMPLE 3 *Let R be the set of all real numbers. Let* $f : R \to R : f(x) = \sin x$ *and* $g : R \to R : g(x) = x^2$. *Prove that* $g \circ f \neq f \circ g$.

SOLUTION Let x be an arbitrary real number. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(\sin x) = (\sin x)^2.$$

$$(f \circ g)(x) = f\{g(x)\} = f(x^2) = \sin x^2.$$

Clearly, $(\sin x)^2 \neq \sin x^2$.

Hence, $g \circ f \neq f \circ g$.

EXAMPLE 4 *Let* $f : R \to R : f(x) = 8x^3$ *and* $g : R \to R : g(x) = x^{1/3}$. *Find* $(g \circ f)$ *and* $(f \circ g)$ *and show that* $g \circ f \neq f \circ g$.

SOLUTION Let $x \in R$. Then, we have

$$(g \circ f)(x) = g\{f(x)\} = g(8x^3) = (8x^3)^{1/3} = 2x.$$

$$(f \circ g)(x) = f\{g(x)\} = f(x^{1/3}) = 8(x^{1/3})^3 = 8x.$$

$\therefore \quad g \circ f \neq f \circ g.$

EXAMPLE 5 *Let* $f : R \to R : f(x) = (x^2 - 3x + 2)$, *find* $(f \circ f)(x)$.

SOLUTION We have

$$(f \circ f)(x) = f\{f(x)\} = f(x^2 - 3x + 2) = f(y), \text{ where } y = (x^2 - 3x + 2)$$
$$= (y^2 - 3y + 2)$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= (x^4 - 6x^3 + 10x^2 - 3x).$$

EXAMPLE 6 *If* $f : R \to R : f(x) = (3 - x^3)^{1/3}$, *show that* $(f \circ f)(x) = x$.

SOLUTION We have

$$(f \circ f)(x) = f\{f(x)\} = f(3 - x^3)^{1/3}$$
$$= f(y), \quad \text{where } y = (3 - x^3)^{1/3}$$
$$= (3 - y^3)^{1/3} = [3 - (3 - x^3)]^{1/3} \quad [\because \quad y^3 = (3 - x^3)]$$
$$= (x^3)^{1/3} = x.$$

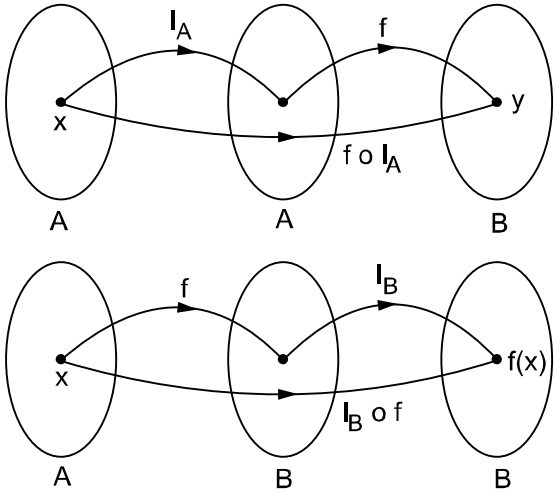
Hence, $(f \circ f)(x) = x$.

EXAMPLE 7 *Let* $f : A \to B$, *and let* I_A *and* I_B *be identity functions on* A *and* B *respectively. Prove that* $(f \circ I_A) = f$ *and* $(I_B \circ f) = f$.

SOLUTION Let $x \in A$ and let $f(x) = y$. Then,

$$(f \circ I_A)(x) = f\{I_A(x)\} = f(x) \qquad [\because \quad I_A(x) = x].$$

$\therefore \quad (f \circ I_A) = f.$



$$\text{And, } (I_B \text{ o } f)(x) = I_B\{f(x)\}$$
$$= I_B(y) \qquad [\because \quad f(x) = y]$$
$$= y \qquad [\because \quad I_B(y) = y]$$
$$= f(x) \qquad [\because \quad y = f(x)].$$
$$\therefore \qquad (I_B \text{ o } f) = f.$$

Hence, $(f \text{ o } I_A) = f$ and $(I_B \text{ o } f) = f$.

EXAMPLE 8 (Associativity) *Let* $f : A \to B$, $g : B \to C$ *and* $h : C \to D$. *Then, prove that* $(h \text{ o } g) \text{ o } f = h \text{ o } (g \text{ o } f)$.

SOLUTION Let $x \in A$. Then,

$$\{(h \text{ o } g) \text{ o } f\}(x) = (h \text{ o } g)\{f(x)\}$$
$$= h[g\{f(x)\}]$$
$$= h[(g \text{ o } f)(x)]$$
$$= \{h \text{ o } (g \text{ o } f)\}(x).$$
$$\therefore \quad (h \text{ o } g) \text{ o } f = h \text{ o } (g \text{ o } f).$$

EXAMPLE 9 *Let* $f : Z \to Z : f(n) = 3n$ *and let* $g : Z \to Z$, *defined by*

$$g(n) = \begin{cases} \frac{n}{3}, & \text{if } n \text{ is a multiple of } 3 \\ 0, & \text{if } n \text{ is not a multiple of } 3. \end{cases}$$

Show that $g \text{ o } f = I_Z$ *and* $f \text{ o } g \neq I_Z$.

SOLUTION Let n be an arbitrary element of Z. Then,

$$(g \text{ o } f)(n) = g\{f(n)\}$$
$$= g(3n) = \frac{3n}{3} = n$$
$$= I_Z(n).$$
$$\therefore \quad (g \text{ o } f) = I_Z.$$

Also, we have

$$(f \text{ o } g)(1) = f\{g(1)\}$$
$$= f(0) \qquad [\because \quad g(1) = 0]$$
$$= (3 \times 0) = 0 \qquad [\because \quad f(n) = 3n].$$
$$I_Z(1) = 1 \ [\because \quad I_Z(x) = x \ \forall \ x \in Z].$$
$$\therefore \quad f \text{ o } g \neq I_Z.$$

EXAMPLE 10 *Let* $A = R - \left\{\frac{3}{5}\right\}$ *and* $B = R - \left\{\frac{7}{5}\right\}$.

Let $f : A \to B : f(x) = \frac{7x + 4}{5x - 3}$ *and* $g : B \to A : g(y) = \frac{3y + 4}{5y - 7}$.

Show that $(g \text{ o } f) = I_A$ *and* $(f \text{ o } g) = I_B$.

SOLUTION Let $x \in A$. Then,

$$(g \circ f)(x) = g[f(x)]$$

$$= g\left(\frac{7x+4}{5x-3}\right)$$

$$= g(y), \text{ where } y = \frac{7x+4}{5x-3} \quad \ldots \text{(i)}$$

$$= \frac{3y+4}{5y-7} = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} \quad \text{[using (i)]}$$

$$= \frac{(21x+12+20x-12)}{(5x-3)} \times \frac{(5x-3)}{(35x+20-35x+21)}$$

$$= \frac{41x}{41} = x = I_A(x).$$

$\therefore \quad (g \circ f) = I_A$.

Again, let $y \in B$. Then,

$$(f \circ g)(y) = f[g(y)]$$

$$= f\left(\frac{3y+4}{5y-7}\right)$$

$$= f(z), \text{ where } z = \frac{3y+4}{5y-7} \quad \ldots \text{(ii)}$$

$$= \frac{7z+4}{5z-3} = \frac{7\left(\frac{3y+4}{5y-7}\right)+4}{5\left(\frac{3y+4}{5y-7}\right)-3}$$

$$= \frac{(21y+28+20y-28)}{(5y-7)} \times \frac{(5y-7)}{(15y+20-15y+21)}$$

$$= \frac{41y}{41} = y = I_B(y).$$

$\therefore \quad (f \circ g) = I_B$.

Hence, $(g \circ f) = I_A$ and $(f \circ g) = I_B$.

EXAMPLE 11 *Let $f : A \to B$ and $g : B \to A$ such that $(g \circ f) = I_A$. Show that f is one-one and g is onto.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow g\{f(x_1)\} = g\{f(x_2)\}$$

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$

$$\Rightarrow I_A(x_1) = I_A(x_2)$$

$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

In order to show that g is onto, let $a \in A$ and let $f(a) = b \in B$.

Then, $g(b) = g[f(a)] = (g \text{ o } f)(a)$

$= I_A(a)$ [$\because$ $g \text{ o } f = I_A$].

Thus, for each $a \in A$, there exists $b \in B$ such that $g(b) = a$.

$\therefore$ g is onto.

EXERCISE 2B

1. Let $A = \{1, 2, 3, 4\}$. Let $f : A \to A$ and $g : A \to A$, defined by $f = \{(1, 4), (2, 1), (3, 3), (4, 2)\}$ and
   $g = \{(1, 3), (2, 1), (3, 2), (4, 4)\}$.
   Find (i) $g \text{ o } f$ (ii) $f \text{ o } g$ (iii) $f \text{ o } f$.
2. Let $f : \{3, 9, 12\} \to \{1, 3, 4\}$ and $g : \{1, 3, 4, 5\} \to \{3, 9\}$ be defined as $f = \{(3, 1), (9, 3), (12, 4)\}$ and
   $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$.
   Find (i) $(g \text{ o } f)$ (ii) $(f \text{ o } g)$.
3. Let $f : R \to R : f(x) = x^2$ and $g : R \to R : g(x) = (x + 1)$.
   Show that $(g \text{ o } f) \neq (f \text{ o } g)$.
4. Let $f : R \to R : f(x) = (2x + 1)$ and $g : R \to R : g(x) = (x^2 - 2)$.
   Write down the formulae for
   (i) $(g \text{ o } f)$ (ii) $(f \text{ o } g)$ (iii) $(f \text{ o } f)$ (iv) $(g \text{ o } g)$.
5. Let $f : R \to R : f(x) = (x^2 + 3x + 1)$ and $g : R \to R : g(x) = (2x - 3)$. Write down the formulae for
   (i) $g \text{ o } f$ (ii) $f \text{ o } g$ (iii) $g \text{ o } g$.
6. Let $f : R \to R : f(x) = |x|$, prove that $f \text{ o } f = f$.
7. Let $f : R \to R : f(x) = x^2$, $g : R \to R : g(x) = \tan x$ and $h : R \to R : h(x) = \log x$.
   Find a formula for $h \text{ o } (g \text{ o } f)$.
   Show that $[h \text{ o } (g \text{ o } f)] \sqrt{\frac{\pi}{4}} = 0$.
8. Let $f : R \to R : f(x) = (2x - 3)$ and $g : R \to R : g(x) = \frac{1}{2}(x + 3)$.
   Show that $(f \text{ o } g) = I_R = (g \text{ o } f)$.
9. Let $f : Z \to Z : f(x) = 2x$. Find $g : Z \to Z : g \text{ o } f = I_Z$.
10. Let $f : N \to N : f(x) = 2x$, $g : N \to N : g(y) = 3y + 4$ and $h : N \to N : h(z) = \sin z$.
    Show that $h \text{ o } (g \text{ o } f) = (h \text{ o } g) \text{ o } f$.

11. If f be a greatest integer function and g be an absolute value function, find the value of $(f \circ g)\left(\frac{-3}{2}\right) + (g \circ f)\left(\frac{4}{3}\right)$. **[CBSE 2007]**

12. Let $f : R \to R : f(x) = x^2 + 2$ and $g : R \to R : g(x) = \frac{x}{x-1}$, $x \neq 1$. Find $f \circ g$ and $g \circ f$ and hence find $(f \circ g)(2)$ and $(g \circ f)(-3)$. **[CBSE 2014]**

ANSWERS (EXERCISE 2B)

1. (i) $(g \circ f) = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
(ii) $(f \circ g) = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$
(iii) $(f \circ f) = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$

2. (i) $(g \circ f) = \{(3, 3), (9, 3), (12, 9)\}$
(ii) $(f \circ g) = \{(1, 1), (3, 1), (4, 3), (5, 3)\}$

4. (i) $(g \circ f)(x) = (4x^2 + 4x - 1)$ (ii) $(f \circ g)(x) = (2x^2 - 3)$
(iii) $(f \circ f)(x) = (4x + 3)$ (iv) $(g \circ g)(x) = (x^4 - 4x^2 + 2)$

5. (i) $(g \circ f)(x) = (2x^2 + 6x - 1)$ (ii) $(f \circ g)(x) = (4x^2 - 6x + 1)$
(iii) $(g \circ g)(x) = (4x - 9)$

7. $[h \circ (g \circ f)](x) = \log(\tan x^2)$ **11.** 2

12. $(f \circ g)(x) = \frac{x^2}{(x-1)^2} + 2$, $(g \circ f)(x) = \frac{x^2+2}{x^2+1}$;

$(f \circ g)(2) = 6$, $(g \circ f)(-3) = \frac{11}{10}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2B)

9. Let $x \in Z$. Then,

$$(g \circ f) = I_Z \Rightarrow (g \circ f)(x) = I_Z(x)$$
$$\Rightarrow g[f(x)] = x$$
$$\Rightarrow g(2x) = x$$
$$\Rightarrow g(y) = \frac{1}{2}y \quad [\text{where } 2x = y].$$

Thus, $g : Z \to Z : g(y) = \frac{1}{2}y$.

11. $(f \circ g)\left(\frac{-3}{2}\right) = f\left\{g\left(\frac{-3}{2}\right)\right\} = f\left(\left|\frac{-3}{2}\right|\right) = f\left(\frac{3}{2}\right) = \left[\frac{3}{2}\right] = 1.$

$(g \circ f)\left(\frac{4}{3}\right) = g\left\{f\left(\frac{4}{3}\right)\right\} = g\left[\frac{4}{3}\right] = g(1) = |1| = 1.$

Required sum = (1 + 1) = 2.

12. $(f \circ g)(x) = f\{g(x)\} = f\left(\frac{x}{x-1}\right) = \frac{x^2}{(x-1)^2} + 2.$

$$(g \circ f)(x) = g\{f(x)\} = g(x^2+2) = \frac{x^2+2}{x^2+2-1} = \frac{x^2+2}{x^2+1}.$$

Invertible Function

Let $f : A \to B$. If there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ then f is called an invertible function and g is called the inverse of f. We write, $f^{-1} = g$.

REMARK Clearly, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Example Let $f : R \to R : f(x) = 2x + 3$.

Let $y = f(x)$. Then,

$$y = f(x) \Rightarrow y = 2x + 3$$
$$\Rightarrow x = \frac{1}{2}(y - 3)$$
$$\Rightarrow f^{-1}(y) = \frac{1}{2}(y - 3) \; [\because y = f(x) \Rightarrow x = f^{-1}(y)].$$

Thus, we define:

$$f^{-1} : R \to R : f^{-1}(y) = \frac{1}{2}(y - 3).$$

THEOREM 1 *If $f : A \to B$ is one-one onto then prove that f is an invertible function.*

PROOF Let $y \in B$. Then, f being one-one onto, there exists a unique $x \in A$ such that $f(x) = y$.

We define $g : B \to A : g(y) = x$. Then,

$$(g \circ f)(x) = g[f(x)]$$
$$= g(y) \quad [\because f(x) = y]$$
$$= x \quad [\because g(y) = x]$$
$$= I_A(x).$$

$\therefore (g \circ f) = I_A$.

$$(f \circ g)(y) = f[g(y)]$$
$$= f(x) \quad [\because g(y) = x]$$
$$= y \quad [\because f(x) = y]$$
$$= I_B(y).$$

$\therefore (f \circ g) = I_B$.

Hence, f is invertible and $f^{-1} = g$.

THEOREM 2 *If $f : A \to B$ is an invertible function, then prove that f is one-one onto.*

PROOF Let $f : A \to B$ be an invertible function. Then, there exists a function $g : B \to A$ such that

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B.$$

Now, $f(x_1) = f(x_2)$

$\Rightarrow\ g\{f(x_1)\} = g\{f(x_2)\}$

$\Rightarrow\ (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow\ I_A(x_1) = I_A(x_2) \qquad [\because\ g \circ f = I_A]$

$\Rightarrow\ x_1 = x_2.$

$\therefore\ f$ is one-one.

Let $y \in B$. Then, $g(y) \in A$. Let $g(y) = x$. Then,

$$g(y) = x$$

$\Rightarrow\ f\{g(y)\} = f(x)$

$\Rightarrow\ (f \circ g)(y) = f(x)$

$\Rightarrow\ I_B(y) = f(x) \qquad [\because\ f \circ g = I_B]$

$\Rightarrow\ y = f(x).$

Thus, for each $y \in B$ there exists $x \in A$ such that $y = f(x)$.

$\therefore\ f$ is onto.

Hence, f is one-one onto.

REMARK f is invertible $\Leftrightarrow$ f is one-one onto.

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : R \to R : f(x) = 4x + 3$ for all $x \in R$. Show that f is invertible and find f^{-1}.*

SOLUTION We have

$$f(x_1) = f(x_2) \ \Rightarrow\ 4x_1 + 3 = 4x_2 + 3$$

$$\Rightarrow\ 4x_1 = 4x_2 \ \Rightarrow\ x_1 = x_2.$$

$\therefore\ f$ is one-one.

Again, $y = 4x + 3 \ \Rightarrow\ x = \dfrac{(y-3)}{4}$.

Now, if $y \in R$ (codomain of f) then there exists $x = \dfrac{(y-3)}{4} \in R$ such that $f(x) = f\left(\dfrac{y-3}{4}\right) = \left\{4 \cdot \dfrac{(y-3)}{4} + 3\right\} = y.$

$\therefore\ f$ is onto.

Thus, f is one-one onto and therefore invertible.

Now, $y = f(x) \ \Rightarrow\ y = 4x + 3$

$$\Rightarrow\ x = \frac{(y-3)}{4}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{(y-3)}{4} \qquad [\because \quad f(x) = y \Rightarrow x = f^{-1}(y)].$$

Thus, we define:

$$f^{-1} : R \to R : f^{-1}(y) = \frac{(y-3)}{4} \text{ for all } y \in R.$$

EXAMPLE 2 *Let R_+ be the set of all positive real numbers. Let $f : R_+ \to [4, \infty[: f(x) = x^2 + 4$. Show that f is invertible and find f^{-1}.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow x_1^2 + 4 = x_2^2 + 4$$
$$\Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1^2 - x_2^2 = 0$$
$$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$
$$\Rightarrow x_1 - x_2 = 0 \qquad [\because \quad (x_1 + x_2) \neq 0]$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Now, $y = x^2 + 4 \Rightarrow x = \sqrt{y-4}$.

For each $y \in [4, \infty[$ there exists $x = \sqrt{y-4}$ in R_+ such that $f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$.

$\therefore$ f is onto.

Thus, f is one-one onto and therefore invertible.

Now, $y = f(x) \Rightarrow y = x^2 + 4$

$$\Rightarrow x = \sqrt{y-4}$$
$$\Rightarrow f^{-1}(y) = \sqrt{y-4}.$$

$\therefore$ $f^{-1} : [4, \infty[\to R_+ : f^{-1}(y) = \sqrt{y-4}$.

EXAMPLE 3 *Let R^+ be the set of all positive real numbers. Let $f : R^+ \to R^+ : f(x) = e^x$ for all $x \in R^+$. Show that f is invertible and hence find f^{-1}.*

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2.$$

Now, for each $y \in R^+$, there exists a positive real number, namely $\log y$ such that

$$f(\log y) = e^{\log y} = y.$$

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

We define:

$$f^{-1} : R^+ \to R^+ : f^{-1}(y) = \log y \text{ for all } y \in R^+.$$

EXAMPLE 4 *Let* $A = \left\{x : x \in R,\ \frac{-\pi}{2} \le x \le \frac{\pi}{2}\right\}$ *and* $B = \{y : y \in R, -1 \le y \le 1\}$. *Show that the function* $f : A \to B : f(x) = \sin x$ *is invertible and hence find* f^{-1}.

SOLUTION Here, $A = \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]$ and $B = [-1,\ 1]$.

Also, $f : A \to B : f(x) = \sin x$.

f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$$

$$\Rightarrow x_1 = x_2 \qquad \left\{\because\ x_1,\ x_2 \in \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]\right\}.$$

$\therefore$ f is one-one.

Also, range $(f) = [-1,\ 1] = B$. So, f is onto.

Thus, f is one-one onto and hence invertible.

Now, $y = f(x) \Rightarrow y = \sin x$

$$\Rightarrow x = \sin^{-1} y$$

$$\Rightarrow f^{-1}(y) = \sin^{-1} y.$$

Thus, we define:

$$f^{-1} : [-1,\ 1] \to \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right] : f^{-1}(y) = \sin^{-1} y.$$

EXAMPLE 5 *Let* $f : N \to Y : f(x) = x^2$, *where* $Y = \text{range}(f)$. *Show that f is invertible and find* f^{-1}.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1^2 - x_2^2 = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \quad [\because\ x_1 + x_2 \neq 0]$$

$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Since range $(f) = Y$, so f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, there exists $x \in N$ such that $f(x) = y$.

Now, $y = f(x) \Rightarrow y = x^2$

$\Rightarrow x = \sqrt{y}$

$\Rightarrow f^{-1}(y) = \sqrt{y}$ [$\because$ $f(x) = y \Rightarrow x = f^{-1}(y)$].

Thus, we define

$f^{-1} : Y \to N : f^{-1}(y) = \sqrt{y}$.

EXAMPLE 6 *Let* $f : [-1, 1] \to Y : f(x) = \dfrac{x}{(x+2)}$, $x \neq -2$ *and* $Y = range\ (f)$. *Show that f is invertible and find* f^{-1}.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1 + 2} = \frac{x_2}{x_2 + 2}$$

$$\Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$$

$$\Rightarrow 2(x_1 - x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Since range $(f) = Y$, so f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, there exists $x \in [-1, 1]$ such that $f(x) = y$.

Now, $y = f(x) \Rightarrow y = \dfrac{x}{(x+2)}$

$$\Rightarrow x = \frac{2y}{(1-y)}$$

$$\Rightarrow f^{-1}(y) = \frac{2y}{(1-y)}.$$

Thus, we define:

$$f^{-1} : [-1, 1] \to Y : f^{-1}(y) = \frac{2y}{(1-y)}, \quad y \neq 1.$$

EXAMPLE 7 *Let* $f : N \to Y : f(x) = 4x^2 + 12x + 15$ *and* $Y = range\ (f)$. *Show that f is invertible and find* f^{-1}. **[CBSE 2013C]**

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$$

$$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$$

$$\Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \quad [\because\ x_1 + x_2 + 3 \neq 0]$$

$$\Rightarrow x_1 = x_2.$$

Also, range $(f) = Y$. So, f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, f being onto, there exists x such that $y = f(x)$.

Now, $y = f(x) \Rightarrow y = 4x^2 + 12x + 15$

$$\Rightarrow y = (2x + 3)^2 + 6$$

$$\Rightarrow (2x + 3) = \sqrt{y - 6}$$

$$\Rightarrow x = \frac{1}{2}(\sqrt{y - 6} - 3)$$

$$\Rightarrow f^{-1}(y) = \frac{1}{2}(\sqrt{y - 6} - 3).$$

Thus, we define:

$$f^{-1} : Y \to N : f^{-1}(y) = \frac{1}{2}(\sqrt{y - 6} - 3).$$

EXAMPLE 8 *Let $f : R \to R : f(x) = 10x + 7$. Find the function $g : R \to R$ such that $g \circ f = f \circ g = I_R$.* **[CBSE 2011]**

SOLUTION Clearly, $g = f^{-1}$ … (i)

Now, $f(x_1) = f(x_2) \Rightarrow 10x_1 + 7 = 10x_2 + 7$

$$\Rightarrow 10x_1 = 10x_2$$

$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Now, $y = f(x) \Rightarrow y = 10x + 7$

$$\Rightarrow x = \frac{(y - 7)}{10}.$$

Clearly, for each $y \in R$ (codomain of f) there exists $x \in R$ such that

$$f(x) = f\left(\frac{y - 7}{10}\right) = \left\{10 \cdot \left(\frac{y - 7}{10}\right) + 7\right\} = y.$$

$\therefore$ f is onto.

Thus, f is one-one onto and therefore, f^{-1} exists.

We define: $f^{-1} : R \to R : f^{-1}(y) = \frac{y - 7}{10}$.

Hence, $g : R \to R : g(y) = \frac{y - 7}{10}$ [using (i)].

EXAMPLE 9 *Let* $f : W \to W : f(n) = \begin{cases} (n - 1), & \text{when } n \text{ is odd} \\ (n + 1), & \text{when } n \text{ is even.} \end{cases}$

Show that f is invertible. Find f^{-1}. **[CBSE 2012]**

SOLUTION Let $f(n_1) = f(n_2)$.

Case 1 *When n_1 is odd and n_2 is even*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 + 1$

$\Rightarrow n_1 - n_2 = 2.$

If n_1 is odd and n_2 is even, then $(n_1 - n_2) \neq 2$.

Thus, we arrive at a contradiction.

So, in this case, $f(n_1) \neq f(n_2)$.

Similarly, when n_1 is even and n_2 is odd, then $f(n_1) \neq f(n_2)$.

Case 2 *When n_1 and n_2 are both odd*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1$

$\Rightarrow n_1 = n_2.$

Case 3 *When n_1 and n_2 are both even*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 + 1 = n_2 + 1$

$\Rightarrow n_1 = n_2.$

Thus, from all the cases, we get $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$.

$\therefore$ f is one-one.

Now, we show that f is onto.

Let $n \in W$.

Case 1 *When n is odd*

In this case, $(n - 1)$ is even

and $f(n - 1) = (n - 1) + 1 = n.$ … (i)

Case 2 *When n is even*

In this case, $(n + 1)$ is odd

and $f(n + 1) = (n + 1) - 1 = n.$ … (ii)

Thus, each $n \in W$ has its pre-image in W.

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

Clearly, we have

$$f^{-1}(n) = \begin{cases} (n-1), & \text{when } n \text{ is odd} \\ (n+1), & \text{when } n \text{ is even} \end{cases}$$ [using (i) and (ii)].

EXAMPLE 10 *Let $A = \{1, 2, 3\}$ and let $f : A \to A$, defined by*
$f = \{(1, 2), (2, 3), (3, 1)\}$.
Find f^{-1}, if it exists.

SOLUTION We have $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$.

Dom $(f) = \{1, 2, 3\} = A$ and range $(f) = \{1, 2, 3\} = A$.

Clearly, different elements in A have different images.

$\therefore$ f is one-one.

Range $(f) = A \Rightarrow f$ is onto.

Thus, f is one-one onto and therefore invertible.

Now, $f(1) = 2,\ \ f(2) = 3$ and $f(3) = 1$

$\Rightarrow\ \ f^{-1}(2) = 1,\ \ f^{-1}(3) = 2$ and $f^{-1}(1) = 3.$

Hence, $f^{-1} = \{(2, 1), (3, 2), (1, 3)\}.$

Some Results on Invertible Functions

THEOREM 1 *Prove that an invertible function has a unique inverse.*

PROOF Let $f : A \to B$, which is one-one onto and therefore, invertible.
If possible, let it have two inverses, say g and h.

Then, $(f \circ g) = I_B$ and $(f \circ h) = I_B$

$\Rightarrow\ (f \circ g)(y) = (f \circ h)(y)$ [each $= I_B(y)$]

$\Rightarrow\ f\{g(y)\} = f\{h(y)\}$ for all $y \in B$

$\Rightarrow\ g(y) = h(y)$ for all $y \in B$ [$\because$ f is one-one].

$\therefore\ g = h.$

Hence, f has a unique inverse.

THEOREM 2 *Let f be an invertible function. Then, prove that*

$$(f^{-1})^{-1} = f.$$

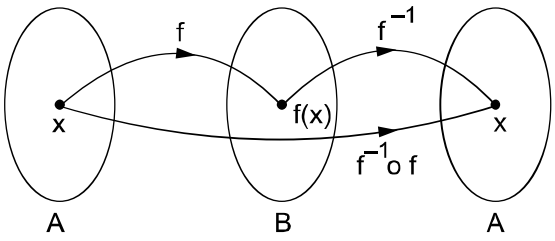
PROOF Let $f : A \to B$, which is invertible.
In order to prove that $(f^{-1})^{-1} = f$, it is sufficient to show that

$f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B.$

Clearly $f : A \to B$ is one-one onto.

$\therefore\ f^{-1} : B \to A$ is one-one onto.

Let $x \in A$ and let $f(x) = y$. Then, $f^{-1}(y) = x.$



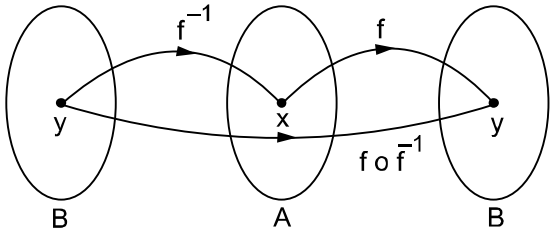
$\therefore\ (f^{-1} \circ f)(x) = f^{-1}\{f(x)\}$

$= f^{-1}(y)$ [$\because$ $f(x) = y$]

$= x$

$= I_A(x).$

$\therefore\ f^{-1} \circ f = I_A.$



Again, let $y \in B.$

Then, f being onto, there exists $x \in A$ such that $f(x) = y$
and therefore, $f^{-1}(y) = x.$

$\therefore \quad (f \text{ o } f^{-1})(y) = f\{f^{-1}(y)\}$

$= f(x) \quad [\because \ f^{-1}(y) = x]$

$= y$

$= I_B(y).$

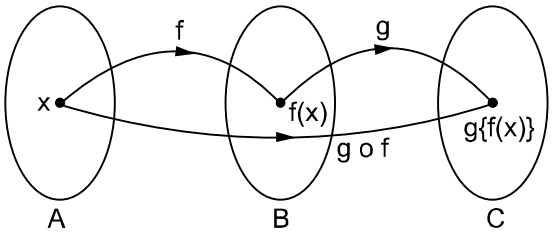
$\therefore \quad f \text{ o } f^{-1} = I_B.$

Thus, $f^{-1} \text{ o } f = I_A$ and $f \text{ o } f^{-1} = I_B.$

Hence, $(f^{-1})^{-1} = f.$

THEOREM 3 *Let $f : A \to B$ and $g : B \to C$ be one-one onto functions. Prove that $(g \text{ o } f) : A \to C$ which is one-one onto and $(g \text{ o } f)^{-1} = f^{-1} \text{ o } g^{-1}$.*

PROOF Let $f : A \to B$ be one-one onto and $g : B \to C$ be one-one onto.



We first show that $g \text{ o } f$ is one-one onto.

$(g \text{ o } f)$ is one-one, since

$(g \text{ o } f)(x_1) = (g \text{ o } f)(x_2)$

$\Rightarrow \quad g\{f(x_1)\} = g\{f(x_2)\}$

$\Rightarrow \quad f(x_1) = f(x_2) \quad [\because \ g \text{ is one-one}]$

$\Rightarrow \quad x_1 = x_2 \quad [\because \ f \text{ is one-one}].$

Let $z \in C$. Then, g being onto, there exists $y \in B$ such that $g(y) = z.$

Now, f being onto, there exists $x \in A$ such that $f(x) = y.$

$\therefore \quad z = g(y)$

$= g\{f(x)\} \quad [\because \ y = f(x)]$

$= (g \text{ o } f)(x).$

Thus, for each $z \in C$, there exists $x \in A$ such that $(g \text{ o } f)(x) = z.$

$\therefore \quad (g \text{ o } f)$ is onto.

Thus, $(g \text{ o } f)$ is one-one onto.

Now, $f(x) = y \ \Rightarrow \ f^{-1}(y) = x.$

And, $g(y) = z \ \Rightarrow \ g^{-1}(z) = y.$

Also, $(g \text{ o } f)(x) = z \ \Rightarrow \ (g \text{ o } f)^{-1}(z) = x.$

$\therefore \quad (f^{-1} \text{ o } g^{-1})(z) = f^{-1}\{g^{-1}(z)\}$

$= f^{-1}(y) \quad [\because \ g^{-1}(z) = y]$

$= x \quad [\because \ f^{-1}(y) = x]$

$= (g \text{ o } f)^{-1}(z).$

Hence, $(g \text{ o } f)^{-1} = (f^{-1} \text{ o } g^{-1}).$

EXERCISE 2C

Very-Short-Answer Questions

1. Prove that the function $f : R \to R : f(x) = 2x$ is one-one and onto.
2. Prove that the function $f : N \to N : f(x) = 3x$ is one-one and into.
3. Show that the function $f : R \to R : f(x) = x^2$ is neither one-one nor onto.
4. Show that the function $f : N \to N : f(x) = x^2$ is one-one and into.
5. Show that the function $f : R \to R : f(x) = x^4$ is neither one-one nor onto.
6. Show that the function $f : Z \to Z : f(x) = x^3$ is one-one and into.
7. Let R_0 be the set of all nonzero real numbers. Then, show that the function $f : R_0 \to R_0 : f(x) = \frac{1}{x}$ is one-one and onto.
8. Show that the function $f : R \to R : f(x) = 1 + x^2$ is many-one into.
9. Let $f : R \to R : f(x) = \frac{2x - 7}{4}$ be an invertible function. Find f^{-1}.

[CBSE 2008C]

10. Let $f : R \to R : f(x) = 10x + 3$. Find f^{-1}.
11. $f : R \to R : f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is rational.} \end{cases}$

    Show that f is many-one and into.
12. Let $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$. Find $(f \text{ o } g)(7)$. **[CBSE 2008]**
13. Let $f : R \to R$ and $g : R \to R$ defined by $f(x) = x^2$ and $g(x) = (x + 1)$. Show that $g \text{ o } f \neq f \text{ o } g$.
14. Let $f : R \to R : f(x) = (3 - x^3)^{1/3}$. Find $f \text{ o } f$. **[CBSE 2010]**
15. Let $f : R \to R : f(x) = 3x + 2$, find $f\{f(x)\}$. **[CBSE 2010C]**
16. Let $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down $g \text{ o } f$. **[CBSE 2014C]**
17. Let $A = \{1, 2, 3, 4\}$ and $f = \{(1, 4), (2,1) (3, 3), (4, 2)\}$. Write down $(f \text{ o } f)$.
18. Let $f(x) = 8x^3$ and $g(x) = x^{1/3}$. Find $g \text{ o } f$ and $f \text{ o } g$.
19. Let $f : R \to R : f(x) = 10x + 7$. Find the function $g : R \to R : g \text{ o } f = f \text{ o } g = I_g$. **[CBSE 2011]**
20. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. State whether f is one-one. **[CBSE 2011]**

ANSWERS (EXERCISE 2C)

9. $f^{-1}(y) = \frac{1}{2}(4y+7)$ for all $y \in R$ **10.** $f^{-1}(y) = \frac{1}{10}(y-3)$ for all $y \in R$

12. 7 **14.** $(f \text{ o } f)(x) = x$ **15.** $f\{f(x)\} = (9x+8)$

16. $(g \text{ o } f) = \{(1, 3), (3, 1), (4, 3)\}$ **17.** $f \text{ o } f = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$

18. $(g \text{ o } f)(x) = 2x$ and $(f \text{ o } g)(x) = 8x$ **19.** $g(x) = \frac{1}{10}(x-7)$ for all $x \in R$

20. Yes

HINTS TO THE GIVEN QUESTIONS (EXERCISE 2C)

1. (i) $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

   (ii) Let $y = 2x$. Then, $x = \frac{1}{2}y$.

   Thus for each y in codomain R, there exists $\frac{1}{2}y$ such that

   $$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

   $\therefore$ f is onto.

2. (i) $f(x_1) = f(x_2) \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

   (ii) If we consider 2 in codomain N, there is no natural number whose image is 2. So, f is into.

3. (i) Clearly, $f(1) = 1^2 = 1$ and $f(-1) = (-1)^2 = 1$.

   So, f is many-one.

   (ii) If we consider -1 in the codomain R then there is no element in R whose square is -1.

   $\therefore$ $-1 \in R$ has no pre-image in R. So, f is many-one into.

4. (i) $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ [$\because$ $x_1, x_2 \in N$].

   $\therefore$ f is one-one.

   (ii) If we consider 2 in the codomain N, then $\sqrt{2} \in N$ and $f(\sqrt{2}) = (\sqrt{2})^2 = 2$.

   So, f is into.

5. (i) $f(1) = 1^4 = 1$ and $f(-1) = (-1)^4 = 1$.

   So, f is many-one.

   (ii) If we consider -1 in the codomain R then there exists no $x \in R$ such that $f(x) = x^4 = -1$. So, f is into.

6. (i) Let $x_1, x_2 \in Z$ and $x_1 \neq x_2$. Then, $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$.

   (ii) Let $2 \in Z$. Then, there exists no $x \in Z$ such that $x^3 = 2$.

   Thus, $2 \in Z$ has no pre-image in Z. So, f is into.

7. (i) $f(x_1) = f(x_2) \Rightarrow \dfrac{1}{x_1} = \dfrac{1}{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

(ii) $y = \dfrac{1}{x} \Rightarrow x = \dfrac{1}{y}$.

Thus, for each y in codomain R_0, there exists $\dfrac{1}{y}$ in domain R_0 such that $f\left(\dfrac{1}{y}\right) = \dfrac{1}{\left(\dfrac{1}{y}\right)} = y$. So, f is onto.

8. (i) $f(-1) = 1 + (-1)^2 = 2$ and $f(1) = (1 + 1^2) = 2$.

So, f is many-one.

(ii) $y = (1 + x^2) \Rightarrow x = \sqrt{y-1}$.

So, when $y < 1$, then $\sqrt{y-1}$ is imaginary.

In particular, $0 \in R$ has no pre-image in R.

$\therefore$ f is into.

9. $y = \dfrac{2x-7}{4} \Rightarrow x = \dfrac{1}{2}(4y + 7)$

$\Rightarrow f^{-1}(y) = \dfrac{4y+7}{2}$ for all $y \in R$.

10. $y = 10x + 3 \Rightarrow x = \dfrac{y-3}{10}$

$\Rightarrow f^{-1}(y) = \dfrac{(y-3)}{10}$.

11. (i) Since all rationals have the same image, namely 1, so f is many-one.

(ii) Range $(f) = \{-1\ 1\} \subset R$. So, f is into.

12. $(f \circ g)(7) = f\{g(7)\} = f(7-7) = f(0) = (0+7) = 7$.

13. $(g \circ f)(x) = g\{f(x)\} = g(x^2) = (x^2 + 1)$.

$(f \circ g)(x) = f\{g(x)\} = f(x+1) = (x+1)^2$.

Hence, $g \circ f \neq f \circ g$.

14. $(f \circ f)(x) = f\{f(x)\} = f\{(3 - x^3)^{1/3}\} = f(y)$, where $y = (3 - x^3)^{1/3}$

$= (3 - y^3)^{1/3} = \{3 - (3 - x^3)\}^{1/3} = (x^3)^{1/3} = x$.

$\therefore$ $(f \circ f)(x) = x$.

15. $f\{f(x)\} = f(3x + 2) = 3(3x + 2) + 2 = (9x + 8)$.

16. Dom $(g \circ f)$ = Dom $(f) = \{1, 3, 4\}$.

$(g \circ f)(1) = g\{f(1)\} = g(2) = 3$.

$(g \circ f)(3) = g\{f(3)\} = g(5) = 1$.

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$.

$\therefore$ $g \circ f = \{(1, 3), (3, 1), (4, 3)\}$.

17. $(f \circ f)(1) = f\{f(1)\} = f(4) = 2$.

$(f \circ f)(2) = f\{f(2)\} = f(1) = 4$.

$(f \circ f)(3) = f\{f(3)\} = f(3) = 3.$

$(f \circ f)(4) = f\{f(4)\} = f(2) = 1.$

$\therefore \quad f \circ f = \{(1, 2), (2, 4), (3, 3), (4, 1)\}.$

18. $(g \circ f)(x) = g[f(x)] = g(8x^3) = g(y)$, where $y = 8x^3$

$= y^{1/3} = (8x^3)^{1/3} = 2x.$

$(f \circ g)(x) = f[g(x)] = f(x^{1/3}) = f(y)$, where $y = x^{1/3}$

$= 8y^3 = 8(x^{1/3})^3 = 8x^{\left(\frac{1}{3} \times 3\right)} = 8x.$

19. $g \circ f = I_R \Rightarrow (g \circ f)(x) = I_R(x) = x$

$\Rightarrow g\{f(x)\} = x$

$\Rightarrow g\{10x + 7\} = x.$

Put $10x + 7 = y$. Then, $x = \dfrac{(y-7)}{10}$.

$\therefore \quad g(y) = \dfrac{(y-7)}{10}.$

Hence, $g : R \to R : g(x) = \dfrac{1}{10}(x-7)$ for all $x \in R$.

20. $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$.

Thus, different elements in A have different images in B.

Hence, f is one-one.

EXERCISE 2D

1. Let $A = \{2, 3, 4, 5\}$ and $B = \{7, 9, 11, 13\}$, and let $f = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$. Show that f is invertible and find f^{-1}.
2. Show that the function $f : R \to R : f(x) = 2x + 3$ is invertible and find f^{-1}.
3. Let $f : Q \to Q : f(x) = 3x - 4$. Show that f is invertible and find f^{-1}.
4. Let $f : R \to R : f(x) = \dfrac{1}{2}(3x + 1)$. Show that f is invertible and find f^{-1}.
5. If $f(x) = \dfrac{(4x + 3)}{(6x - 4)}$, $x \neq \dfrac{2}{3}$, show that $(f \circ f)(x) = x$ for all $x \neq \dfrac{2}{3}$. Hence, find f^{-1}.
6. Show that the function f on $A = R - \left\{\dfrac{2}{3}\right\}$, defined as $f(x) = \dfrac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1}. **[CBSE 2013]**
7. Show that the function f on $A = R - \left\{\dfrac{-4}{3}\right\}$ into itself, defined by $f(x) = \dfrac{4x}{(3x+4)}$ is one-one and onto. Hence, find f^{-1}.

8. Let R_+ be the set of all positive real numbers. Show that the function $f : R_+ \to [-5, \infty[: f(x) = (9x^2 + 6x - 5)$ is invertible. Find f^{-1}.

9. Let $f : N \to R : f(x) = 4x^2 + 12x + 15$. Show that $f : N \to$ range (f) is invertible. Find f^{-1}. **[CBSE 2010, '13C]**

10. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B : f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Hence, find f^{-1}. **[CBSE 2013]**

11. Let f and g be two functions from R into R, defined by $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Find f o g and g o f. **[CBSE 2014C]**

ANSWERS (EXERCISE 2D)

1. $f^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$ **2.** $f^{-1}(y) = \dfrac{1}{2}(y - 3)$

3. $f^{-1}(y) = \dfrac{1}{3}(y + 4)$ **4.** $f^{-1}(y) = \dfrac{(2y-1)}{3}$

5. $f^{-1}(y) = \dfrac{(4y + 3)}{(6y - 4)}$ **6.** $f^{-1}(y) = \dfrac{(4y + 3)}{6y - 4}$

7. $f^{-1}(y) = \dfrac{(4y)}{(4 - 3y)}$ **8.** $f^{-1}(y) = \dfrac{\sqrt{y+6} - 1}{3}$

9. $f^{-1}(y) = \dfrac{\sqrt{y-6} - 3}{2}$ **10.** $f^{-1}(y) = \dfrac{2y-1}{y-1}$

11. $(f \text{ o } g)(x) = |x| - x$ and $(g \text{ o } f)(x) = |x| - x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2D)

1. Clearly $f(2) = 7, f(3) = 9, f(4) = 11$ and $f(5) = 13$.
   Thus, different elements in A have different images in B.
   So, f is one-one.
   Range $(f) = \{7, 9, 11, 13\} = B$. So, f is onto.
   $f^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$.

2. $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
   $\therefore$ f is one-one.
   If $y \in R$ then there exists $x = \dfrac{y-3}{2} \in R$ such that
   $$f(x) = f\left(\frac{y-3}{2}\right) = \left\{2 \cdot \frac{(y-3)}{2} + 3\right\} = y.$$
   $\therefore$ f is onto.

$$y = f(x) \Rightarrow y = 2x + 3$$
$$\Rightarrow x = \frac{1}{2}(y - 3) \Rightarrow f^{-1}(y) = \frac{1}{2}(y - 3).$$

5. $(f \circ f)(x) = f\{f(x)\} = f\left\{\frac{4x + 3}{6x - 4}\right\} = \frac{4\left(\frac{4x + 3}{6x - 4}\right) + 3}{6\left(\frac{4x + 3}{6x - 4}\right) - 4}$

$$= \frac{(16x + 12 + 18x - 12)}{(24x + 18 - 24x + 16)} = \frac{34x}{34} = x = I(x).$$

$\therefore \quad f \circ f = I \Rightarrow f^{-1} = f.$

6. $f(x_1) = f(x_2) \Rightarrow \frac{4x_1 + 3}{6x_1 - 4} = \frac{4x_2 + 3}{6x_2 - 4}$

$$\Rightarrow (4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$$
$$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$$
$$\Rightarrow 34x_1 = 34x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Let y be an arbitrary element of A. Then,

$$f(x) = y \Rightarrow \frac{4x + 3}{6x - 4} = y$$
$$\Rightarrow (6x - 4)y = 4x + 3 \Rightarrow 6xy - 4y = 4x + 3$$
$$\Rightarrow 6xy - 4x = 4y + 3 \Rightarrow x(6y - 4) = (4y + 3)$$
$$\Rightarrow x = \frac{4y + 3}{6y - 4}.$$

Thus, for each $y \in A$, there exists $x = \frac{4y + 3}{6y - 4}$ such that

$$f(x) = f\left(\frac{4y + 3}{6y - 4}\right) = \frac{4 \cdot \left(\frac{4y + 3}{6y - 4}\right) + 3}{6 \cdot \left(\frac{4y + 3}{6y - 4}\right) - 4}$$

$$= \frac{16y + 12 + 18y - 12}{24y + 18 - 24y + 16} = \frac{34y}{34} = y.$$

$\therefore$ f is onto.

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$.

$\therefore \quad f^{-1}(y) = \frac{4y + 3}{6y - 4}$, for all $y \in A$.

7. $f(x_1) = f(x_2) \Rightarrow \frac{4x_1}{3x_1 + 4} = \frac{4x_2}{3x_2 + 4} \Rightarrow \frac{x_1}{3x_1 + 4} = \frac{x_2}{3x_2 + 4}$

$$\Rightarrow x_1(3x_2 + 4) = x_2(3x_1 + 4)$$
$$\Rightarrow 3x_1x_2 + 4x_1 = 3x_1x_2 + 4x_2$$
$$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Let y be an arbitrary element of A. Then,

$$f(x) = y \Rightarrow \frac{4x}{3x+4} = y$$
$$\Rightarrow 3xy + 4y = 4x \Rightarrow 4x - 3xy = 4y$$
$$\Rightarrow x(4 - 3y) = 4y \Rightarrow x = \frac{4y}{(4-3y)}.$$

Thus, for each $y \in A$, there exists an $x \in A$ such that

$$f(x) = f\left(\frac{4y}{4-3y}\right) = \frac{4 \cdot \left(\frac{4y}{4-3y}\right)}{3 \cdot \left(\frac{4y}{4-3y}\right) + 4}$$
$$= \frac{16y}{12y + 16 - 12y} = \frac{16y}{16} = y.$$

$\therefore$ f is onto.

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$.

$\therefore$ $f^{-1}(y) = \frac{4y}{4-3y}$.

8. $y = (3x+1)^2 - 6 \Rightarrow x = \frac{\sqrt{y+6} - 1}{3} \Rightarrow f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$.

9. $f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$
$$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0 \Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$$
$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

Since $f : N \to \text{range}(f)$, so f is onto.

Now $4x^2 + 12x + 15 = y \Rightarrow (2x+3)^2 + 6 = y$

$$\Rightarrow (2x+3) = \sqrt{y-6} \Rightarrow x = \frac{\sqrt{y-6} - 3}{2}.$$

$\therefore$ $f^{-1}(y) = \frac{\sqrt{y-6} - 3}{2}$.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $f : N \to N : f(x) = 2x$ is
   (a) one-one and onto (b) one-one and into
   (c) many-one and onto (d) many-one and into
2. $f : N \to N : f(x) = x^2 + x + 1$ is
   (a) one-one and onto (b) one-one and into
   (c) many-one and onto (d) many-one and into

3. $f : R \to R : f(x) = x^2$ is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

4. $f : R \to R : f(x) = x^3$ is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

5. $f : R^+ \to R^+ : f(x) = e^x$ is

(a) many-one and into (b) many-one and onto

(c) one-one and into (d) one-one and onto

6. $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1] : f(x) = \sin x$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

7. $f : R \to R : f(x) = \cos x$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

8. $f : C \to R : f(z) = |z|$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

9. Let $A = R - \{3\}$ and $B = R - \{1\}$. Then, $f : A \to B : f(x) = \frac{(x-2)}{(x-3)}$ is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

10. Let $f : N \to N : f(n) = \begin{cases} \frac{1}{2}(n+1), \text{ when } n \text{ is odd} \\ \frac{n}{2}, \text{ when } n \text{ is even.} \end{cases}$

Then, f is

(a) one-one and into (b) one-one and onto

(c) many-one and into (d) many-one and onto

11. Let A and B be two non-empty sets and let $f : (A \times B) \to (B \times A) : f(a, b) = (b, a)$. Then, f is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

12. Let $f : Q \to Q : f(x) = (2x + 3)$. Then, $f^{-1}(y) = ?$

(a) $(2y - 3)$ (b) $\frac{1}{(2y-3)}$ (c) $\frac{1}{2}(y - 3)$ (d) none of these

13. Let $f : R - \left\{\frac{-4}{3}\right\} \to R - \left\{\frac{4}{3}\right\} : f(x) = \frac{4x}{(3x+4)}$. Then, $f^{-1}(y) = ?$

(a) $\frac{4y}{(4-3y)}$ (b) $\frac{4y}{(4y+3)}$ (c) $\frac{4y}{(3y-4)}$ (d) none of these

14. Let $f : N \to X : f(x) = 4x^2 + 12x + 15$. Then, $f^{-1}(y) = ?$

(a) $\frac{1}{2}(\sqrt{y-4}+3)$ (b) $\frac{1}{2}(\sqrt{y-6}-3)$

(c) $\frac{1}{2}(\sqrt{y-4}+5)$ (d) none of these

15. If $f(x) = \frac{(4x+3)}{(6x-4)}$, $x \neq \frac{2}{3}$ then $(f \text{ o } f)(x) = ?$

(a) x (b) $(2x-3)$ (c) $\frac{4x-6}{3x+4}$ (d) none of these

16. If $f(x) = (x^2 - 1)$ and $g(x) = (2x+3)$ then $(g \text{ o } f)(x) = ?$

(a) $(2x^2+3)$ (b) $(3x^2+2)$ (c) $(2x^2+1)$ (d) none of these

17. If $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ then $f(x) = ?$

(a) x^2 (b) (x^2-1) (c) (x^2-2) (d) none of these

18. If $f(x) = \frac{1}{(1-x)}$ then $(f \text{ o } f \text{ o } f)(x) = ?$

(a) $\frac{1}{(1-3x)}$ (b) $\frac{x}{(1+3x)}$ (c) x (d) none of these

19. If $f(x) = \sqrt[3]{3 - x^3}$ then $(f \text{ o } f)(x) = ?$

(a) $x^{1/3}$ (b) x (c) $(1 - x^{1/3})$ (d) none of these

20. If $f(x) = x^2 - 3x + 2$ then $(f \text{ o } f)(x) = ?$

(a) x^4 (b) $x^4 - 6x^3$ (c) $x^4 - 6x^3 + 10x^2$ (d) none of these

21. If $f(x) = 8x^3$ and $g(x) = x^{1/3}$ then $(g \text{ o } f)(x) = ?$

(a) x (b) $2x$ (c) $\frac{x}{2}$ (d) $3x^2$

22. If $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$ then $\{h \text{ o } (g \text{ o } f)\}\left(\sqrt{\frac{\pi}{4}}\right) = ?$

(a) 0 (b) 1 (c) $\frac{1}{x}$ (d) $\frac{1}{2}\log\frac{\pi}{4}$

23. If $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$ then $(g \text{ o } f) = ?$

(a) {(3, 1), (1, 3), (3, 4)} (b) {(1, 3), (3, 1), (4, 3)}

(c) {(3, 4), (4, 3), (1, 3)} (d) {(2, 5), (5, 2), (1, 5)}

24. Let $f(x) = \sqrt{9 - x^2}$. Then, dom $(f) = ?$

(a) $[-3, 3]$ (b) $(-\infty, -3]$

(c) $[3, \infty)$ (d) $(-\infty, -3] \cup (4, \infty)$

25. Let $f(x) \sqrt{\dfrac{x-1}{x-4}}$ · Then, dom $(f) = ?$

(a) $[1, 4)$ (b) $[1, 4]$

(c) $(-\infty, 4]$ (d) $(-\infty, 1] \cup (4, \infty)$

26. Let $f(x) = e^{\sqrt{x^2 - 1}} \cdot \log (x - 1)$. Then, dom $(f) = ?$

(a) $(-\infty, 1]$ (b) $[-1, \infty)$

(c) $(1, \infty)$ (d) $(-\infty, -1] \cup (1, \infty)$

27. Let $f(x) = \dfrac{x}{(x^2 - 1)}$ · Then, dom $(f) = ?$

(a) R (b) $R - \{1\}$ (c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

28. Let $f(x) = \dfrac{\sin^{-1} x}{x} \cdot$ Then, dom $(f) = ?$

(a) $(-1, 1)$ (b) $[-1, 1]$ (c) $[-1, 1] - \{0\}$ (d) none of these

29. Let $f(x) = \cos^{-1} 2x$. Then, dom $(f) = ?$

(a) $[-1, 1]$ (b) $\left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$ (c) $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ (d) $\left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$

30. Let $f(x) = \cos^{-1}(3x - 1)$. Then, dom $(f) = ?$

(a) $\left(0, \dfrac{2}{3}\right)$ (b) $\left[0, \dfrac{2}{3}\right]$ (c) $\left[\dfrac{-2}{3}, \dfrac{2}{3}\right]$ (d) none of these

31. Let $f(x) = \sqrt{\cos x}$. Then, dom $(f) = ?$

(a) $\left[0, \dfrac{\pi}{2}\right]$ (b) $\left[\dfrac{3\pi}{2}, 2\pi\right]$

(c) $\left[0, \dfrac{\pi}{2}\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$ (d) none of these

32. Let $f(x) = \sqrt{\log (2x - x^2)}$. Then, dom $(f) = ?$

(a) $(0, 2)$ (b) $[1, 2]$ (c) $(-\infty, 1]$ (d) none of these

33. Let $f(x) = x^2$. Then, dom (f) and range (f) are respectively

(a) R and R (b) R^+ and R^+ (c) R and R^+ (d) R and $R - \{0\}$

34. Let $f(x) = x^3$. Then, dom (f) and range (f) are respectively

(a) R and R (b) R^+ and R^+

(c) R and R^+ (d) R^+ and R

35. Let $f(x) = \log(1 - x) + \sqrt{x^2 - 1}$. Then, dom $(f) = ?$

(a) $(1, \infty)$ (b) $(-\infty, -1]$ (c) $[-1, 1)$ (d) $(0, 1)$

36. Let $f(x) = \dfrac{1}{(1 - x^2)}$. Then, range $(f) = ?$

(a) $(-\infty, 1]$ (b) $[1, \infty)$ (c) $[-1, 1]$ (d) none of these

37. Let $f(x) = \dfrac{x^2}{(1 + x^2)}$. Then, range $(f) = ?$

(a) $[1, \infty)$ (b) $[0, 1)$ (c) $[-1, 1]$ (d) $(0, 1]$

38. The range of $f(x) = x + \dfrac{1}{x}$ is

(a) $[-2, 2]$ (b) $[2, \infty)$ (c) $(-\infty, -2]$ (d) none of these

39. The range of $f(x) = a^x$, where $a > 0$ is

(a) $]-\infty, 0]$ (b) $]-\infty, 0)$ (c) $[0, \infty)$ (d) $(0, \infty)$

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) **2.** (b) **3.** (d) **4.** (a) **5.** (d) **6.** (b) **7.** (c) **8** (c) **9.** (b) **10.** (d)
11. (a) **12.** (c) **13.** (a) **14.** (b) **15.** (a) **16.** (c) **17.** (c) **18.** (c) **19.** (b) **20.** (d)
21. (b) **22.** (a) **23.** (b) **24.** (a) **25.** (d) **26.** (c) **27.** (d) **28.** (c) **29.** (b) **30.** (b)
31. (c) **32.** (d) **33.** (c) **34.** (a) **35.** (b) **36.** (b) **37.** (b) **38.** (d) **39.** (d)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. $2x = 3 \Rightarrow x = \dfrac{3}{2} \notin N$. So, f is into.

2. $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$\Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$.

$\therefore$ f is one-one.

$f(x) = 1 \Rightarrow x^2 + x + 1 = 1 \Rightarrow x(x + 1) = 0 \Rightarrow x = 0$ or $x = -1$.

And, none of 0 and –1 is in N. So, f is into.

5. $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

For each $x \in R^+$ $\exists \log x \in R^+$ s.t. $f(\log x) = x$.

So, f is onto.

7. $\cos(2\pi - \theta) = \cos\theta \Rightarrow f$ is many-one.

Range $(f) = [-1, 1] \subset R \Rightarrow f$ is into.

8. $i \neq -i$. But $f(i) = f(-i) = 1$. So, f is many-one.

$-1 \in R$ having no pre-image in C. So, f is into.

9. $f(x_1) = f(x_2) \Rightarrow \dfrac{(x_1 - 2)}{(x_1 - 3)} = \dfrac{(x_2 - 2)}{(x_3 - 3)} \Rightarrow x_1 = x_2$. So, f is one-one.

Let $\dfrac{x-2}{x-3} = y$. Then, $x = \dfrac{3y-2}{y-1}$. Clearly, $y \neq 1$ and $x \neq 3$.

$\therefore$ $f(x) = y$ and so f is onto.

10. $f(1) = f(2)$ shows that f is many-one.

If n is odd then $(2n - 1)$ is odd and $f(2n - 1) = n$.

If n is even then $2n$ is even and $f(2n) = n$.

$\therefore$ f is onto.

12. $y = 2x + 3 \Rightarrow x = \dfrac{1}{2}(y - 3) \Rightarrow f^{-1}(y) = \dfrac{1}{2}(y - 3)$.

13. $y = \dfrac{4x}{3x+4} \Rightarrow x = \dfrac{4y}{(4 - 3y)} \Rightarrow f^{-1}(y) = \dfrac{4y}{(4 - 3y)}$.

14. $y = 4x^2 + 12x + 15 = (2x + 3)^2 + 6 \Rightarrow x = \dfrac{1}{2}(\sqrt{y - 6} - 3)$

$\therefore$ $f^{-1}(y) = \dfrac{1}{2}(\sqrt{y - 6} - 3)$.

15. $f(x) = \dfrac{4x + 3}{6x - 4} = y$ (say).

$$\text{Then } f(y) = \frac{4y + 3}{6y - 4} = \frac{4\left(\dfrac{4x + 3}{6x - 4}\right) + 3}{6\left(\dfrac{4x + 3}{6x - 4}\right) - 4} = \frac{34x}{34} = x.$$

$\Rightarrow$ $f[f(x)] = x \Rightarrow (f \circ f)(x) = x$.

16. $(g \circ f)(x) = g[f(x)] = g(x^2 - 1)$

$= 2(x^2 - 1) + 3 = (2x^2 + 1)$.

17. Let $x + \dfrac{1}{x} = z$. Then,

$$f(z) = f\left(x + \frac{1}{x}\right) = \left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = (z^2 - 2).$$

$\Rightarrow$ $f(x) = (x^2 - 2)$.

18. $(f \circ f)(x) = f\{f(x)\} = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{\left(1 - \dfrac{1}{1-x}\right)} = \dfrac{1-x}{-x} = \dfrac{x-1}{x}$

$\Rightarrow$ $\{f \circ (f \circ f)\}(x) = f\{(f \circ f)(x)\} = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1 - \dfrac{x-1}{x}} = x$.

19. $(f \circ f)(x) = f\{f(x)\} - \{(3 - x^3)^{1/3}\} = f(y)$, where $y = (3 - x^3)^{1/3}$

$= (3 - y^3)^{1/3} = [3 - \{3 - x^3\}]^{1/3} = (x^3)^{1/3} = x$.

20. $(f \circ f)(x) = f\{f(x)\} = f(x^2 - 3x + 2) = f(y)$, where $y = (x^2 - 3x + 2)$

$= y^2 - 3y + 2 = (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= (x^4 - 6x^3 + 10x^2 - 3x)$.

21. $(g \circ f)(x) = g[f(x)] = g(8x^3) = (8x^3)^{1/3} = 2x$.

22. $\{h \circ (g \circ f)\}(x) = (h \circ g)\{f(x)\} = (h \circ g)(x^2)$

$= h\{g(x^2)\} = h(\tan x^2) = \log(\tan x^2)$.

$\therefore \quad \{h \circ (g \circ f)\}\sqrt{\frac{\pi}{4}} = \log\left(\tan\frac{\pi}{4}\right) = \log 1 = 0.$

23. $\text{Dom}(g \circ f) = \text{dom}(f) = \{1, 3, 4\}$.

$(g \circ f)(1) = g\{f(1)\} = g(2) = 3, (g \circ f)(3) = g\{f(3)\} = g(5) = 1$

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$

$\therefore \quad g \circ f = \{(1,3), (3,1), (4,3)\}$.

24. $f(x)$ is defined only when $9 - x^2 \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow -3 \leq x \leq 3$.

$\therefore \quad \text{dom}(f) = [-3, 3]$.

25. $f(x)$ is defined when $x - 4 \neq 0$ and $\frac{x-1}{x-4} \geq 0$

$\Rightarrow \quad x \neq 4$ and $(x \geq 4$ or $x \leq 1) \Rightarrow (x > 4$ or $x \leq 1)$

$\Rightarrow \quad \text{dom}(f) = (-\infty, 1] \cup (4, \infty)$.

26. $f(x)$ is defined only when $(x^2 - 1) \geq 0$ and $(x - 1) > 0$

$\Rightarrow \quad (x-1)(x+1) \geq 0$ and $(x-1) > 0 \Rightarrow x + 1 \geq 0$ and $x - 1 > 0 \Rightarrow x > 1$

$\therefore \quad \text{dom}(f) = (1, \infty)$.

27. $f(x)$ is not defined when $(x^2 - 1) = 0$, i.e., when $(x-1)(x+1) = 0$,

i.e., when $x = 1$ or $x = -1$.

$\therefore \quad \text{dom}(f) = R - \{1, -1\}$.

28. $\frac{\sin^{-1} x}{x}$ is defined only when $x \neq 0$ and $x \in [-1, 1]$.

$\therefore \quad \text{dom}(f) = [-1, 1] - \{0\}$.

29. $\sin^{-1} 2x$ is defined only when $2x \in [-1, 1] \Rightarrow x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$.

30. $\cos^{-1}(3x - 1)$ is defined only when $(3x - 1) \in [-1, 1]$

$\Rightarrow \quad 3x \in [0, 2] \Rightarrow x \in \left[0, \frac{2}{3}\right] \Rightarrow \text{dom}(f) = \left[0, \frac{2}{3}\right]$.

31. $f(x)$ is defined only when $\cos x \geq 0$

$\Rightarrow \quad x$ lies in 1st or 4th quadrant

$\Rightarrow \quad \text{dom}(f) = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$.

32. $f(x)$ is defined only when $\log(2x - x^2) \geq 0$

$\Rightarrow\ (2x - x^2) \geq 1 \Rightarrow\ (1 + x^2 - 2x) \leq 0 \Rightarrow\ (1 - x)^2 \leq 0 \Rightarrow\ (1 - x) = 0 \Rightarrow\ x = 1.$

$\therefore\ \text{dom}(f) = \{1\}.$

33. $f(x) = x^2$ is clearly defined for each $x \in R$. So, $\text{dom}(f) = R$.

$y = x^2 \Rightarrow\ x = \pm\sqrt{y}.$

When $y < 0$, there is no real value of x. So, $y \geq 0$.

$\therefore\ \text{range}(f) = R^+.$

34. $f(x) = x^3$ is defined for each $x \in R$. So, $\text{dom}(f) = R$.

For each $y \in R$, $y^{1/3} \in R$ and so $x = y^{1/3}$ is real.

$\therefore\ \text{range}(f) = R.$

35. Let $f(x) = g(x) + h(x)$, where $g(x) = \log(1 - x)$ and $h(x) = \sqrt{x^2 - 1}$.

$g(x)$ is defined only when $1 - x > 0 \Rightarrow\ x < 1$. So, $\text{dom}(g) = (-\infty, 1)$.

$h(x)$ is defined only when $x^2 - 1 \geq 0 \Rightarrow\ x \geq 1$ or $x \leq -1$.

$\therefore\ \text{dom}(h) = (-\infty, -1] \cup [1, \infty).$

$\therefore\ \text{dom}(f) = \text{dom}(g) \cap \text{dom}(h) = (-\infty, -1].$

36. $y = \dfrac{1}{(1 - x^2)} \Rightarrow\ x = \sqrt{1 - \dfrac{1}{y}}.$

Clearly, x is not defined when $y = 0$ or $1 - \dfrac{1}{y} < 0$, i.e., $y = 0$ or $y < 1$.

$\therefore\ \text{range}(f) = [1, \infty).$

37. $y = \dfrac{x^2}{(1 + x^2)} \Rightarrow\ x = \sqrt{\dfrac{y}{1 - y}}.$

Clearly, x is defined only when $\dfrac{y}{(1 - y)} \geq 0$, and $(1 - y) \neq 0$, i.e., when $0 \leq y < 1$.

So, $\text{range}(f) = [0, 1).$

38. $y = \dfrac{x^2 + 1}{x} \Rightarrow\ x^2 - xy + 1 = 0 \Rightarrow\ x = \dfrac{y \pm \sqrt{y^2 - 4}}{2}.$

x is defined when $(y^2 - 4) \geq 0 \Rightarrow\ y^2 \geq 4 \Rightarrow\ y \geq 2$ or $y \leq -2$.

$\therefore\ \text{range}(f) = (-\infty, -2] \cup [2, \infty).$

39. Clearly, $a^x > 0$ whatever may be the value of x.

$\therefore\ \text{range}(f) = (0, \infty).$

3. BINARY OPERATIONS

CLOSURE PROPERTY An operation $*$ on a nonempty set S is said to satisfy the closure property, if

$$a \in S,\ b \in S \ \Rightarrow\ a * b \in S \text{ for all } a, b \in S.$$

Also, in this case, we say that S is closed for $*$.

An operation $*$ on a nonempty set S, satisfying the closure property is known as a *binary operation*.

EXAMPLE 1 (i) Addition on the set N of all natural numbers is a binary operation, since

$$a \in N,\ b \in N \ \Rightarrow\ a + b \in N \text{ for all } a, b \in N.$$

(ii) Multiplication on N is a binary operation, since

$$a \in N,\ b \in N \ \Rightarrow\ a \times b \in N \text{ for all } a, b \in N.$$

Similarly, addition as well as multiplication is a binary operation on each one of the sets Z, Q, R and C of integers, rationals, reals and complex numbers respectively.

EXAMPLE 2 Let S be a nonempty set and $P(S)$ be its power set. Then, the union operation on $P(S)$ is a binary operation, since

$$A \in P(S),\ B \in P(S) \ \Rightarrow\ A \cup B \in P(S) \text{ for all } A, B \in P(S).$$

Similarly, intersection on $P(S)$ is a binary operation, as

$$A \in P(S),\ B \in P(S) \ \Rightarrow\ A \cup B \in P(S) \text{ for all } A, B \in P(S).$$

EXAMPLE 3 Subtraction on N is not a binary operation, since

$$3 \in N, 5 \in N \text{ but } (3 - 5) = -2 \notin N.$$

Subtraction on the set Z of all integers is a binary operation, since

$$a \in Z,\ b \in Z \ \Rightarrow\ a - b \in Z \text{ for all } a,\ b \in Z.$$

EXAMPLE 4 Addition on the set S of all irrationals is not a binary operation, since

$$2 + \sqrt{3} \in S,\ 2 - \sqrt{3} \in S \text{ but } (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4 \notin S.$$

Multiplication on the set S of all irrationals is not a binary operation, since

$$\sqrt{2} \in S,\ -\sqrt{2} \in S \text{ but } (\sqrt{2})(-\sqrt{2}) = -2 \notin S.$$

EXAMPLE 5 Let N be the set of all natural numbers. Then, the exponential operation $(a,\ b) \Rightarrow a^b$ is a binary operation on N, since

$$a \in N,\ b \in N \ \Rightarrow\ a^b \in N \text{ for all } a,\ b \in N.$$

Let Z be the set of all integers. The exponential operation $(a, b) \to a^b$ is not a binary operation on Z, since $2 \in Z$, $-3 \in Z$ but $2^{-3} = \frac{1}{2^3} = \frac{1}{8} \notin Z$.

Properties of a binary operation

(i) **Associative law** A binary operation $*$ on a nonempty set S is said to be associative, if

$$(a * b) * c = a * (b * c) \text{ for all } a, b, c \in S.$$

(ii) **Commutative law** A binary operation * on a nonempty set S is said to be commutative, if

$$a * b = b * a \text{ for all } a, b \in S.$$

(iii) **Distributive law** Let $*$ and $\circ$ be two binary operations on a nonempty set S. We say that $*$ is distributive over $\circ$, if

$$a * (b \circ c) = (a * b) \circ (a * c) \text{ for all } a, b, c \in S.$$

EXAMPLE 1 Let R be the set of all real numbers. Then,

(i) addition on R satisfies the closure property, the associative law and the commutative law,

(ii) multiplication on R satisfies the closure property, the associative law and the commutative law,

(iii) multiplication distributes addition on R, since

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ for all } a, b, c \in R.$$

EXAMPLE 2 Let Z be the set of all integers. Then, subtraction on Z is clearly a binary operation. But, it is neither commutative nor associative, as

$$(3 - 5) \neq (5 - 3) \text{ and } (3 - 4) - 5 \neq 3 - (4 - 5).$$

IDENTITY ELEMENT Let $*$ be a binary operation on a nonempty set S. An element $e \in S$, if it exists such that

$$a * e = e * a = a \text{ for all } a \in S,$$

is called an identity element of S, with respect to $*$.

EXAMPLE 1 (i) For addition on R, zero is the identity element in R, since

$$a + 0 = 0 + a = a \text{ for all } a \in R.$$

(ii) For multiplication on R, 1 is the identity element in R, since

$$a \times 1 = 1 \times a = a \text{ for all } a \in R.$$

EXAMPLE 2 Let $P(S)$ be the power set of a nonempty set S. Then, ϕ is the identity element for union on $P(S)$ as

$$A \cup \phi = \phi \cup A = A \text{ for all } A \in P(S).$$

Also, S is the identity element for intersection on $P(S)$, since

$$A \cap S = S \cap A = A \text{ for all } A \in P(S).$$

INVERSE OF AN ELEMENT Let $*$ be a binary operation on a nonempty set S and let e be the identity element.

Let $a \in S$. We say that a is invertible if there exists an element $b \in S$ such that

$$a * b = b * a = e.$$

Also, in this case, b is called the inverse of a, and we write, $a^{-1} = b$.

EXAMPLE 1 *Consider addition on Z.*

Clearly, the additive identity is 0, since

$$a + 0 = 0 + a = a \text{ for all } a \in Z.$$

Also, corresponding to each $a \in Z$, there exists $-a \in Z$ such that

$$a + (-a) = (-a) + a = 0.$$

Thus, the *additive inverse* of a is $-a$.

EXAMPLE 2 *Consider multiplication on Z.*

Clearly, 1 is the multiplicative identity on Z, since

$$a \times 1 = 1 \times a = a \text{ for all } a \in Z.$$

Since $1 \times 1 = 1$, so the multiplicative inverse of 1 is 1.

Since $(-1) \times (-1) = 1$, so the multiplicative inverse of (-1) is (-1).

No integer other than 1 and -1 has its multiplicative inverse in Z.

SOLVED EXAMPLES

EXAMPLE 1 *Let Z be the set of all integers. Then, addition on Z satisfies the following properties:*

(i) *Closure Property*

We know that the sum of two integers is always an integer, i.e., $a \in Z,\ b \in Z \Rightarrow a + b \in Z$ for all $a, b \in Z$.

(ii) *Associative Law*

For all $a, b, c \in Z$, we have

$$(a + b) + c = a + (b + c).$$

(iii) *Commutative Law*

For all $a, b \in Z$, we have

$$a + b = b + a.$$

(iv) *Existence of Additive Identity*

Clearly, $0 \in Z$ is the additive identity, since

$$0 + a = a + 0 = a \text{ for all } a \in Z.$$

(v) *Existence of Additive Inverse*

For each $a \in Z$, there exists $-a \in Z$ such that

$$a + (-a) = (-a) + a = 0.$$

So, $-a$ is the additive inverse of a.

EXAMPLE 2 *Let R_0 be the set of all nonzero real numbers. Then, multiplication on R_0 satisfies the following properties:*

(i) *Closure Property*
We know that the product of two nonzero real numbers is a nonzero real number.

Thus, $a \in R_0,\ b \in R_0 \Rightarrow ab \in R_0$ for all $a, b \in R_0$.

(ii) *Associative Law*
For all $a, b, c \in R_0$, we have $(ab)c = a(bc)$.

(iii) *Commutative Law*
For all $a, b \in R_0$, we have $ab = ba$.

(iv) *Existence of Multiplicative Identity*
Clearly, $1 \in R_0$ is the multiplicative identity, since

$$1 \times a = a \times 1 \text{ for all } a \in R_0.$$

(v) *Existence of Multiplicative Inverse*
For each $a \in R_0$ there exists $\frac{1}{a} \in R_0$ such that

$$a \times \frac{1}{a} = \frac{1}{a} \times a = 1.$$

Thus, the multiplicative inverse of a is $\frac{1}{a}$.

EXAMPLE 3 *Show that the operation $*$ on Z, defined by*

$$a * b = a + b + 1 \text{ for all } a, b \in Z$$

satisfies (i) the closure property, (ii) the associative law and (iii) the commutative law.
(iv) Find the identity element in Z.
(v) What is the inverse of an element $a \in Z$?

SOLUTION (i) *Closure Property*

Let $a \in Z,\ b \in Z$. Then,

$$a * b = a + b + 1.$$

Now, $a \in Z,\ b \in Z \Rightarrow a + b \in Z$

$$\Rightarrow a + b + 1 \in Z.$$

$\therefore$ $*$ on Z satisfies the closure property.

(ii) *Associative Law*

For all $a, b, c \in Z$, we have:

$$\begin{aligned}(a * b) * c &= (a + b + 1) * c \\ &= (a + b + 1) + c + 1 \\ &= a + b + c + 2.\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * (b + c + 1) \\ &= a + (b + c + 1) + 1 \\ &= a + b + c + 2.\end{aligned}$$

$\therefore$ $(a * b) * c = a * (b * c)$.

(iii) *Commutative Law*

For all $a, b \in Z$, we have

$$\begin{aligned} a * b &= a + b + 1 \\ &= b + a + 1 \qquad [\because \ a + b = b + a] \\ &= b * a. \end{aligned}$$

(iv) *Existence of Identity Element*

Let e be the identity element in Z.

Then, $a * e = a \ \Rightarrow \ a + e + 1 = a$

$\Rightarrow \ e = -1.$

Thus, $-1 \in Z$ is the identity element for $*$.

(v) *Existence of Inverse*

Let $a \in Z$ and let its inverse be b. Then,

$$\begin{aligned} a * b = -1 &\Rightarrow \ a + b + 1 = -1 \\ &\Rightarrow \ b = -(2 + a). \end{aligned}$$

Clearly, $2 \in Z, \ a \in Z \ \Rightarrow \ -(2 + a) \in Z.$

Thus, each $a \in Z$ has $-(2 + a) \in Z$ as its inverse.

EXAMPLE 4 *Show that the operation $*$ on $Q - \{1\}$, defined by*

$$a * b = a + b - ab \text{ for all } a, \ b \in Q - \{1\}$$

satisfies (i) the closure property, (ii) the associative law, (iii) the commutative law.

(iv) What is the identity element?

(v) For each $a \in Q - \{1\}$, find the inverse of a.

SOLUTION (i) *Closure Property*

Let $a \in Q - \{1\}$ and $b \in Q - \{1\}$.

We know that Q is closed for addition, subtraction and multiplication.

$\therefore \ a + b - ab \in Q.$

But, $a * b = 1 \Rightarrow \ a + b - ab = 1$

$\Rightarrow \ a(1 - b) = (1 - b) \ \Rightarrow \ a = 1,$

which is a contradiction since $1 \notin Q - \{1\}$.

$\therefore \ a * b \neq 1.$

Thus, $a \in Q - \{1\}, b \in Q - \{1\} \ \Rightarrow \ a * b \in Q - \{1\}.$

$\therefore$ $*$ is a binary operation on $Q - \{1\}$.

(ii) *Associative Law*

Let $a, \ b, \ c \in Q - \{1\}$. Then,

$$\begin{aligned} (a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - (a + b - ab)c \\ &= (a + b + c) - (ab + bc + ac) + abc. \end{aligned}$$

And, $a * (b * c) = a * (b + c - bc)$

$$\begin{aligned} &= a + (b + c - bc) - a(b + c - bc) \\ &= (a + b + c) - (ab + bc + ac) + abc. \end{aligned}$$

$\therefore \quad (a * b) * c = a * (b * c)$.

Hence, $*$ is associative.

(iii) *Commutative Law*

Let $a, b \in Q - \{1\}$. Then,

$$\begin{aligned} a * b &= a + b - ab \\ &= b + a - ba \quad [\because + \text{ and } \cdot \text{ are commutative on } Q - \{1\}] \\ &= b * a. \end{aligned}$$

Hence, $*$ is commutative.

(iv) *Existence of Identity Element*

Let e be the identity element.

Then, for all $a \in Q - \{1\}$, we have

$$\begin{aligned} a * e = a &\Rightarrow a + e - ae = a \\ &\Rightarrow e(1 - a) = 0 \Rightarrow e = 0 \in Q - \{1\}. \end{aligned}$$

Now, $a * 0 = a + 0 - a \times 0 = a$.

And, $0 * a = 0 + a - 0 \times a = a$.

Thus, 0 is the identity element in $Q - \{1\}$.

(v) *Existence of Inverse*

Let $a \in Q - \{1\}$ and let $a^{-1} = b$. Then,

$$\begin{aligned} a * b = 0 &\Rightarrow a + b - ab = 0 \\ &\Rightarrow a = ab - b = (a - 1)b \\ &\Rightarrow b = \frac{a}{(a-1)} \in Q - \{1\}. \end{aligned}$$

$$\therefore \quad a^{-1} = \frac{a}{(a-1)} \in Q - \{1\}.$$

Thus, each $a \in Q - \{1\}$ has its inverse in $Q - \{1\}$.

EXAMPLE 5 *On the set N of all natural numbers, define the operation $*$ on N by*

$$m * n = \gcd(m, n) \text{ for all } m, n \in N.$$

Show that $*$ is commutative as well as associative.

SOLUTION (i) *Commutativity*

For all $m, n \in N$, we have $\gcd(m, n) = \gcd(n, m)$.

$\therefore \quad m * n = n * m \; \forall \; m, n \in N$.

Hence, $*$ is commutative on N.

(ii) *Associativity*

Let $m, n, p \in N$. Then,

$$\begin{aligned} (m * n) * p &= [\gcd(m, n)] * p \\ &= \gcd[\gcd\{(m, n), p\}] \\ &= \gcd[\{m, \gcd(n, p)\}] \end{aligned}$$

[$\because$ gcd of three numbers = gcd {(gcd of any two, third)}]

$$= \gcd(m, n * p) = m * (n * p).$$

Hence, $*$ is associative on N.

EXAMPLE 6 *Consider the set* $A = \{-1, 1\}$ *with multiplication operation. We may prepare its composition table as shown below.*

$\times$	1	–1
1	1	–1
–1	–1	1

Multiplication on A satisfies the following properties:

(i) *Closure Property*
Since all the entries of the composition table are in A, so A is closed for multiplication.

(ii) *Associative Law*
Since multiplication of integers is associative, in particular, multiplication on A is associative.

(iii) *Commutative Law*
Since every row of the table coincides with the corresponding column,

i.e., 1st row coincides with 1st column,

2nd row coincides with 2nd column.

So, multiplication is commutative on A.

(iv) *Existence of Identity*
Clearly, 1 is the identity element in A, since
$1 \times 1 = 1$ and $(-1) \times 1 = 1 \times (-1) = -1$.

(v) *Existence of Inverse*
It is clear from the table that
$1 \times 1 = 1 \Rightarrow$ inverse of 1 is 1,
$(-1) \times (-1) = 1 \Rightarrow$ inverse of (–1) is (–1).

NEW OPERATIONS

EXAMPLE 7 *Let* $A = \{1, 2, 3, 4, 5\}$. *Define an operation* $\wedge$ *by*
$a \wedge b = \min \{a, b\}$.
Then, we may prepare its composition table as given below.

$\vee$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Closure Property

Since all the entries of the composition table are from given set A, so A is closed for the operation $\wedge$.

Commutative Law

In the given table every row coincides with the corresponding column,

i.e., 1st row coincides with 1st column,
2nd row coincides with 2nd column, and so on.

$\therefore$ $\wedge$ on A is commutative.

EXAMPLE 8 *Let* $A = \{1, 2, 3, 4, 5\}$. *Define an operation* $\vee$ *by* $a \vee b = \max\{a, b\}$.
Prepare its composition table.
Show that A is closed for the given operation and that the given operation is commutative.

SOLUTION We prepare the table as given below.

$\vee$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

Closure Property

Since all the entries of the composition table are from the given set, so closure property is satisfied.

Commutative Law

Clearly, every row coincides with the corresponding column. So, commutative law is satisfied.

EXERCISE 3A

Very-Short-Answer Questions

1. Let * be a binary operation on the set I of all integers, defined by $a * b = 3a + 4b - 2$. Find the value of $4 * 5$. **[CBSE 2011C]**
2. The binary operation * on R is defined by $a * b = 2a + b$. Find $(2 * 3) * 4$. **[CBSE 2012]**
3. Let $*$ be a binary operation on the set of all nonzero real numbers, defined by $a * b = \frac{ab}{5}$. Find the value of x given that $2 * (x * 5) = 10$. **[CBSE 2014]**
4. Let $*: R \times R \to R$ be a binary operation given by $a * b = a + 4b^2$. Then, compute $(-5) * (2 * 0)$. **[CBSE 2014C]**

5. Let $*$ be a binary operation on the set Q of all rational numbers given as $a * b = (2a - b)^2$ for all $a, b \in Q$. Find $3 * 5$ and $5 * 3$. Is $3 * 5 = 5 * 3$?

[CBSE 2008C]

6. Let $*$ be a binary operation on N given by $a * b =$ lcm of a and b. Find the value of $20 * 16$.

Is $*$ (i) commutative, (ii) associative? **[CBSE 2008C]**

7. If $*$ be the binary operation on the set Z of all integers defined by $a * b = (a + 3b^2)$, find $2 * 4$. **[CBSE 2009]**

8. Show that $*$ on Z^+ defined by $a * b = |a - b|$ is not a binary operation.

9. Let $*$ be a binary operation on N, defined by $a * b = a^b$ for all $a, b \in N$. Show that $*$ is neither commutative nor associative.

10. Let $a * b =$ lcm (a, b) for all values of $a, b \in N$.
    (i) Find $(12 * 16)$.
    (ii) Show that $*$ is commutative on N.
    (iii) Find the identity element in N.
    (iv) Find all invertible elements in N.

11. Let Q^+ be the set of all positive rational numbers.
    (i) Show that the operation $*$ on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.
    (ii) Show that $*$ is commutative.
    (iii) Show that $*$ is not associative. **[CBSE 2008]**

12. Show that the set $A = \{-1, 0, 1\}$ is not closed for addition.

13. Show that $*$ on $R - \{-1\}$, defined by $(a * b) = \frac{a}{(b+1)}$ is neither commutative nor associative. **[CBSE 2007]**

14. For all $a, b \in R$, we define $a * b = |a - b|$.
    Show that $*$ is commutative but not associative.

15. For all $a, b \in N$, we define $a * b = a^3 + b^3$.
    Show that $*$ is commutative but not associative.

16. Let X be a nonempty set and $*$ be a binary operation on $P(X)$, the power set of X, defined by $A * B = A \cap B$ for all $A, B \in P(X)$.
    (i) Find the identity element in $P(X)$.
    (ii) Show that X is the only invertible element in $P(X)$.

17. A binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ is defined as

$$a * b = \begin{cases} a + b; & \text{if } a + b < 6 \\ a + b - 6; & \text{if } a + b \geq 6. \end{cases}$$

Show that 0 is the identity for this operation and each element a has an inverse $(6 - a)$. **[CBSE 2011C]**

ANSWERS (EXERCISE 3A)

1. 30 **2.** 18 **3.** $x = 25$ **4.** 11
5. 1, 49; No **6.** $20 * 16 = 80$; (i) Yes (ii) Yes **7.** 50
10. (i) 48 (iii) 1 (iv) 1 **16.** (i) X

HINTS TO THE GIVEN QUESTIONS (EXERCISE 3A)

1. $4 * 5 = 3 \times 4 + 4 \times 5 - 2 = 12 + 20 - 2 = 30.$
2. $2 * 3 = 2 \times 2 + 3 = 4 + 3 = 7.$
   $\therefore \quad (2 * 3) * 4 = 7 * 4 = 2 \times 7 + 4 = 14 + 4 = 18.$
3. $2 * (x * 5) = 10 \Rightarrow 2 * \left(\frac{x \times 5}{5}\right) = 10$
   $\Rightarrow 2 * x = 10 \Rightarrow \frac{2 \times x}{5} = 10$
   $\Rightarrow 2x = 50 \Rightarrow x = 25.$
4. $(2 * 0) = 2 + 4 \times 0^2 = 2 + 4 \times 0 = 2 + 0 = 2.$
   $\therefore \quad (-5) * (2 * 0) = (-5) * 2 = (-5) + 4 \times 2^2 = (-5) + 16 = 11.$
5. $(3 * 5) = (2 \times 3 - 5)^2 = (6 - 5)^2 = 1^2 = 1.$
   $(5 * 3) = (2 \times 5 - 3)^2 = (10 - 3)^2 = 7^2 = 49.$
   $\therefore \quad (3 * 5) \neq (5 * 3).$
6. $(20 * 16) = \text{lcm}\ (20, 16) = 80.$
   (i) For all $a, b \in N$, we know that $\text{lcm}\ \{a, b\} = \{b, a\}$. So, $a * b = b * a.$
   $\therefore$ * is commutative on N.
   (ii) For all $a, b, c \in N$, we know that
   $\text{lcm}\ \{a, b \text{ and } c\} = \text{lcm}\ \{(a, b) \text{ and } c\}.$
   $\therefore$ * is associative on N.
7. $(2 * 4) = (2 + 3 \times 4^2) = (2 + 3 \times 16) = (2 + 48) = 50.$
8. Let a be an arbitrary positive integer. Then,
   $a * a = |a - a| = 0$, which is not a positive integer.
   $\therefore$ * on Z^+ is not a binary operation.
9. (i) $2 * 5 = 2^5 = 32$ and $5 * 2 = 5^2 = 25.$
   Hence, $2 * 5 \neq 5 * 2.$
   (ii) $(2 * 1) * 4 = 2^1 * 4 = (2 * 4) = 2^4 = 16.$
   $2 * (1 * 4) = 2 * 1^4 = 2 * 1 = 2^1 = 2.$
   $\therefore \quad (2 * 1) * 4 \neq 2 * (1 * 4).$
10. (i) $12 * 16 = \text{lcm}\ (12, 16) = (4 \times 3 \times 4) = 48.$
    (ii) $a * b = \text{lcm}(a, b) = \text{lcm}(b, a) = b * a.$
    (iii) Let e be the identity element in N.

4	12, 16
	3, 4

Then, for all $a \in N$, we have

$a * e = a \Rightarrow \text{lcm}(a, e) = a \Rightarrow e = 1.$

$\therefore$ 1 is the identity element in N.

(iv) Let $a \in N$ be an arbitrary element.

Then, $a * b = 1 \Rightarrow \text{lcm}(a, b) = 1 \Rightarrow a = b = 1.$

$\therefore$ 1 is the only element in N, which is invertible.

11. (i) $a \in Q^+,\ b \in Q^+ \Rightarrow a + b \in Q^+$

$\Rightarrow \frac{1}{2}(a + b) \in Q^+$

$\Rightarrow a * b \in Q^+.$

(ii) $a * b = \frac{1}{2}(a + b) = \frac{1}{2}(b + a) = b * a.$

(iii) We have

$(3 * 4) * 5 = \frac{1}{2}(3 + 4) * 5 = \left(\frac{7}{2} * 5\right) = \frac{1}{2}\left(\frac{7}{2} + 5\right) = \frac{17}{4}.$

$3 * (4 * 5) = \left(3 * \frac{9}{2}\right) = \frac{1}{2}\left(3 + \frac{9}{2}\right) = \frac{15}{4}.$

$\therefore (3 * 4) * 5 \neq 3 * (4 * 5).$

12. $-1 \in A$ and $-1 \in A$. But $(-1) + (-1) = -2 \notin A$.

13. (i) $2 * 3 = \frac{2}{(3 + 1)} = \frac{2}{4} = \frac{1}{2}$ and $3 * 2 = \frac{3}{(2 + 1)} = \frac{3}{3} = 1.$

$\therefore 2 * 3 \neq 3 * 2$. So, * is not commutative.

(ii) $(2 * 3) * 1 = \frac{1}{2} * 1 = \frac{1/2}{(1 + 1)} = \frac{1}{4}.$

$(3 * 3) = \frac{3}{(1 + 1)} = \frac{3}{2}.$

$\therefore 2 * (3 * 1) = 2 * \frac{3}{2} = \frac{2}{\left(\frac{3}{2} + 1\right)} = \left(2 \times \frac{2}{5}\right) = \frac{4}{5}.$

Thus, $(2 * 3) * 1 \neq 2 * (3 * 1).$

14. (i) For all $a, b \in R$ we have

$a * b = |a - b| = |-(a - b)| = |b - a| = (b * a).$

$\therefore$ * is commutative.

(ii) We have

$(2 * 3) * 4 = |2 - 3| * 4 = |-1| * 4 = 1 * 4 = |1 - 4| = |-3| = 3.$

$2 * (3 * 4) = 2 * |3 - 4| = 2 * |-1| = 2 * 1 = |2 - 1| = |1| = 1.$

$\therefore (2 * 3) * 4 \neq 2 * (3 * 4).$

Hence, * is not associative.

15. (i) For all $a, b \in N$ we have

$a * b = a^3 + b^3 = b^3 + a^3 = b * a.$

$\therefore$ * is commutative.

(ii) $(1 * 2) * 3 = (1^3 + 2^3) * 3 = (9 * 3) = (9^3 + 3^3)$

$= (729 + 27) = 756.$

$$1 * (2 * 3) = 1 * (2^3 + 3^3) = 1 * (8 + 27) = 1 * 35$$
$$= \{1^3 + (35)^3\}.$$

$\therefore \quad (1 * 2) * 3 \neq 1 * (2 * 3)$.

16. (i) Since $A \cap X = A$ for all A in $P(X)$.

$\therefore \quad X$ is the identity element.

(ii) Let A be invertible in $P(X)$ and let B be its inverse.

Then, $A \cap B = X$.

This is possible only when $A = B = X$.

$\therefore \quad X$ is the only invertible element in $P(X)$ and its inverse is X.

17. $a * 0 = a + 0 = a \qquad [\because \quad 0 \leq a \leq 5]$.

So, 0 is the identity.

$a * (6 - a) = a + (6 - a) - 6 = 0 \quad [\because \quad a + (6 - a) \geq 6]$.

$\therefore \quad$ The inverse of a is $(6 - a)$.

EXERCISE 3B

1. Define $*$ on N by $m * n = \text{lcm } (m, n)$.
   Show that $*$ is a binary operation which is commutative as well as associative.
2. Define $*$ on Z by $a * b = a - b + ab$.
   Show that $*$ is a binary operation on Z which is neither commutative nor associative.
3. Define $*$ on Z by $a * b = a + b - ab$.
   Show that $*$ is a binary operation on Z which is commutative as well as associative.
4. Consider a binary operation on $Q - \{1\}$, defined by
   $$a * b = a + b - ab.$$
   (i) Find the identity element in $Q - \{1\}$.
   (ii) Show that each $a \in Q - \{1\}$ has its inverse.
5. Let Q_0 be the set of all nonzero rational numbers. Let $*$ be a binary operation on Q_0, defined by $a * b = \frac{ab}{4}$ for all $a, b \in Q_0$.
   (i) Show that $*$ is commutative and associative.
   (ii) Find the identity element in Q_0.
   (iii) Find the inverse of an element a in Q_0.
6. On the set Q^+ of all positive rational numbers, define an operation $*$ on Q^+ by $a * b = \frac{ab}{2}$ for all $a, b \in Q^+$.
   Show that
   (i) $*$ is a binary operation on Q^+,

(ii) $*$ is commutative,
(iii) $*$ is associative.
Find the identity element in Q^+ for $*$.
What is the inverse of $a \in Q^+$?

7. Let Q^+ be the set of all positive rational numbers.
   (i) Show that the operation $*$ on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.
   (ii) Show that $*$ is commutative.
   (iii) Show that $*$ is not associative.
8. Let Q be the set of all rational numbers. Define an operation $*$ on $Q - \{-1\}$ by $a * b = a + b + ab$.
   Show that
   (i) $*$ is a binary operation on $Q - \{-1\}$,
   (ii) $*$ is commutative,
   (iii) $*$ is associative,
   (iv) zero is the identity element in $Q - \{-1\}$ for $*$,
   (v) $a^{-1} = \left(\frac{-a}{1+a}\right)$, where $a \in Q - \{-1\}$.
9. Let $A = N \times N$. Define $*$ on A by
   $$(a, b) * (c, d) = (a + c, b + d).$$
   Show that
   (i) A is closed for $*$,
   (ii) $*$ is commutative,
   (iii) $*$ is associative,
   (iv) identity element does not exist in A.
10. Let $A = \{1, -1, i, -i\}$ be the set of four 4th roots of unity. Prepare the composition table for multiplication on A and show that
    (i) A is closed for multiplication,
    (ii) multiplication is associative on A,
    (iii) multiplication is commutative on A,
    (iv) 1 is the multiplicative identity,
    (v) every element in A has its multiplicative inverse.

ANSWERS (EXERCISE 3B)

4. (i) 0 (ii) $a^{-1} = \frac{a}{(a-1)}$ **5.** (ii) 4 (iii) $a^{-1} = \frac{16}{a}$ **6.** $e = 2, a^{-1} = \frac{4}{a}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 3B)

1. We know that the lcm of two natural numbers is a natural number.
   So, N is closed for $*$.

We know that $\text{lcm}\,(m, n) = \text{lcm}\,(n, m)$. So, $m * n = n * m$.

$\text{lcm}\,(m, n, p) = \text{lcm}\,\{\text{lcm}\,(m, n), p\} = \text{lcm}\,\{m, \text{lcm}\,(n, p)\}$.

2. $a \in Z, b \in Z \Rightarrow (a - b) \in Z$ and $ab \in Z$

$\Rightarrow \{(a - b) + ab\} \in Z \Rightarrow a - b + ab \in Z$.

$\therefore$ Z is closed for $*$.

Show that $3 * 2 \neq 2 * 3$ and $(4 * 3) * 2 \neq 4 * (3 * 2)$.

4. (i) Let e be the identity element. Then, for all $a \in Q - \{1\}$, we have

$$a * e = a \Rightarrow a + e - ae = a$$

$$\Rightarrow e(1 - a) = 0 \Rightarrow e = 0 \quad [\because a \neq 1].$$

$\therefore$ 0 is the identity element.

(ii) Let $a \in Q - \{1\}$ be an arbitrary element and let b be its inverse.

Then, $a * b = 0 \Rightarrow a + b - ab = 0 \Rightarrow ab - b = a$

$$\Rightarrow b(a - 1) = a \Rightarrow b = \frac{a}{(a - 1)}.$$

Thus, each $a \in Q - \{1\}$ has $\dfrac{a}{(a - 1)}$ as its inverse.

5. (i) For all $a, b, c \in Q_0$, we have

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a.$$

$$\text{And, } (a * b) * c = \frac{ab}{4} * c = \frac{\frac{ab}{4} \cdot c}{4} = \frac{(ab)c}{16}.$$

$$\text{Also, } a * (b * c) = a * \frac{bc}{4} = \frac{a\left(\frac{bc}{4}\right)}{4} = \frac{a(bc)}{16}.$$

But, $(ab)c = a(bc)$. Hence, $(a * b) * c = a * (b * c)$.

(ii) Let e be the identity element and let $a \in Q_0$. Then,

$$a * e = a \Rightarrow \frac{ae}{4} = a \Rightarrow e = 4.$$

$\therefore$ 4 is the identity element in Q_0.

(iii) Let $a \in Q_0$ and let its inverse be b. Then,

$$a * b = e \Rightarrow \frac{ab}{4} = 4 \Rightarrow b = \frac{16}{a} \in Q_0.$$

Thus, each $a \in Q_0$ has $\dfrac{16}{a}$ as its inverse.

6. $a * e = a \Rightarrow \dfrac{ae}{2} = a \Rightarrow e = 2.$

$a * b = 2 \Rightarrow \dfrac{ab}{2} = 2 \Rightarrow b = \dfrac{4}{a} \Rightarrow a^{-1} = \dfrac{4}{a}.$

9. (i) Let $(a, b) \in A$ and $(c, d) \in A$. Then, $a, b, c, d \in N$.

$(a, b) * (c, d) = (a + c, b + d) \in A \quad [\because a + c \in N, b + d \in N]$.

$\therefore$ A is closed for $*$.

(ii) $(a, b) * (c, d) = (a + c, b + d)$

$= (c + a, d + b)$ [$\because a + c = c + a$ and $b + d = d + b$]

$= (c, d) * (a, b)$.

(iii) $[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f)$

$= [(a + c) + e, (b + d) + f]$

$= [a + (c + e), b + (d + f)]$

$= (a, b) * (c + e, d + f)$

$= (a, b) * [(c, d) * (e, f)]$.

(iv) $(a, b) * (0, 0) = (a + 0, b + 0) = (a, b)$.

But, $(0, 0) \notin A$, since $0 \notin N$.

So, identity element does not belong to A.

10. We may prepare the composition table for multiplication on A as given below:

$\times$	1	–1	i	$-i$
1	1	–1	i	$-i$
–1	–1	1	$-i$	i
i	i	$-i$	–1	1
$-i$	$-i$	i	1	–1

Clearly, 1 is the identity element.

$(1)^{-1} = 1$, $(-1)^{-1} = -1$, $(i)^{-1} = -i$ and $(-i)^{-1} = i$.